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Preface

Markovian jump systems typically consist of a finite number of subsystems and a jumping law governing the active/deactivate mode switches among these subsystems. The subsystems are usually modeled as differential/difference equations, and the jumping law is a continuous-time/discrete-time Markov chain. Markovian jump systems are a powerful modeling tool in many engineering areas. For instance, abrupt changes are often seen in practical systems, due to the abrupt environmental disturbances, the component and interconnection failures, the abrupt changes of the operation point for the nonlinear plant, etc. The system can be modeled as having different dynamics before and after the abrupt changes, and the changes are usually memoryless and thus Markovian, hence resulting in a Markovian jump system. Indeed, Markovian jump systems can often be seen in the study of networked control systems, circuit and power systems, flight control systems, robotic systems, and so on, where the stability analysis, tracking, fault-tolerant control, etc., have been extensively discussed. However, the theoretical development of Markovian jump systems has its own challenges, mainly due to the exclusive Markovian jumping law. It is well-known that the whole system can still be unstable even if all the subsystems are stable, while the whole system can be stable even if all the subsystems are unstable. Furthermore, the existence of random noises, delays, nonlinearity, modeling error and disturbance, robust stability, H_∞ control and filtering, adaptive control, practical stability and optimal control, etc. are also important topics in Markovian jump systems.

This book discusses the stability analysis of different Markovian jump systems as well as some applications. With multiple stability definitions, we analyze and design Markovian jump systems in a systematic manner. This book is written primarily for postgraduate students in control theory and applications, and can also be useful for the researchers and engineers in this area. In order to use this book, the reader should have the basic knowledge on linear control theory, matrix analysis, optimization techniques, probability and stochastic processes.

This book contains seven chapters. A brief description of each chapter goes as follows. Chapter 1 introduces the related history and background of Markovian jump systems as well as the necessary definitions and notations. Chapter 2 deals with the robust stability and H_∞ control issues for a class of uncertain Markovian jump systems with delays. Chapter 3 investigates various stochastic stability criteria for nonlinear Markovian jump systems with asynchronous switching and extended asynchronous switching. Chapter 4 discusses a robust adaptive control scheme for a class of nonlinear uncertain Markovian jump systems with nonlinear state-dependent uncertainty. Chapter 5 studies the practical stability in probability, practical stability in the pth mean, and the practical controllability for stochastic nonlinear Markovian jump systems. Chapter 6 considers the Markovian jump system model for networked control systems. Chapter 7 discusses two applications based on the Markov jump theory, i.e., the fault-tolerant control for wheeled mobile manipulators and the jump linear quadratic regulator problem.

We hope the reader will find this book useful.

Hefei, China
May 2017

Yu Kang

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Contents

Acronyms and Notations

$\in$	Belongs to
$\forall$	For all
$\mathbb{R}$	The set of real values
$\mathbb{R}_+$	The set of all nonnegative real numbers
$\mathbb{R}^n$	n-dimensional real Euclidean space
$\mathbb{Z}$	The set of all the integers
lim	Limit
max	Maximum
min	Minimum
sup	Supremum
inf	Infimum
$x \wedge y$	The smaller number in x and y
I	Identity matrix
$\lambda_i(A)$	The ith eigenvalues of matrix A
A^T	Transpose of matrix A
A^{-1}	Inverse of matrix A
$tr(A)$	Trace of matrix A
$det(A)$	Determinant of matrix A
$A \otimes B$	Kronecker product of matrices A and B
$\lambda_{\max}(A)$	Maximum eigenvalue of matrix A
$\lambda_{\min}(A)$	Minimum singular value of matrix A
$P > 0 (P \geq 0)$	Matrix P is positive (semi-)definite
$P < 0 (P \leq 0)$	Matrix P is negative (semi-)definite
$diag\{A_1, \ldots, A_m\}$	Block diagonal matrix with blocks $\{A_1, \ldots, A_m\}$
$\|\cdot\|$	Euclidean norm or induced norm
$\|\cdot\|_F$	F-norm, i.e. $\|\|X\|\|_F = X^T F X$
∇f	The gradient vector
$(\Omega, \mathscr{F}, P)$	Probability space
$\mathscr{F}_t^W$	Minimal σ-algebra generated by the random process $(W_s,\ s \leq t)$

$\mathscr{F}_t^\theta$	A class of $\mathscr{F}_t^W$ where θ denotes the Markov process
$\sigma(x)$	Minimal σ-algebra generated by x
$E(\cdot)$	The expectation
$B(t)$	Brownian motion
$r(t)$	Continuous-time Markov process
$\Pi = [\pi_{ij}]$	Transition rate matrix of $r(t)$
$\mathscr{S}$	A finite regime space, $\mathscr{S} = \{1, 2, \ldots, N\}$
τ_k	Discrete-time Markov chain
$\Lambda = [\lambda_{ij}]$	Transition probability matrix of τ_k
$\mathscr{M}$	The regime space of τ_k, $\mathscr{M} = \{0, 1, 2, \ldots, \bar{\tau}, \infty\}$
$\mathscr{A}$	Action space
$\mathfrak{L}$	Infinitesimal generator
$\mathscr{K}$	A function α: $\mathbb{R}_+ \to \mathbb{R}_+$, where α is continuous, strictly increasing, and $\alpha(0) = 0$
$\mathscr{K}_\infty$	The set of all the unbounded functions in $\mathscr{K}$
$\mathscr{K}\mathscr{L}$	A function β: $\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, where $\beta(\cdot, t)$ is of class $\mathscr{K}$ in the first argument for each fixed $t \geq 0$ and $\beta(s, t)$ decreases to 0 as $t \to +\infty$ for each fixed $s \geq 0$
$\mathscr{C}^i([-\mu, 0]; \mathbb{R}^n)$	The set of i-th continuously differentiable functions over $[-\mu, 0] \to \mathbb{R}^n$
$\mathscr{C}^b_{\mathscr{F}_0}([-\mu, 0]; \mathbb{R}^n)$	The set of bounded $\mathscr{F}_t$-measurable random variables with domain $\mathscr{C}([-\mu, 0]; \mathbb{R}^n)$

Chapter 1
Introduction to Markovian Jump Systems

This chapter first introduces the basic concepts of MJSs, and then some research topics including the robust stochastic stability, the imprecise jumping parameters, the nonlinear Markovian jump systems, the practical stability, etc. Notations, necessary definitions and useful lemmas are also given.

1.1 Background

With the fast development and wide applications of the information technology, a large number of man-made systems are emerging in various areas including communications, aeronautics, integrated manufactures, transport management, etc. These systems are featured by the distinct state driven by the discrete events, and are essentially complex due to the nonlinear, stochastic and emergent behaviours. This thus means that the theories of conventional Continuous Variable Dynamic Systems (CVDS) are not directly applicable, and the theories of Discrete Event Dynamic Systems (DEDS) developed in 1980's become the effective methodology for such complex systems. Furthermore, with the emergence of various complex systems due to the developments of the large-scale parallel computers, global communications and high accuracy manufactures, etc., the two aforementioned types of systems interact with each other and further form the so-called Hybrid Dynamic Systems (HDSs). This new type of systems now become key to the modern information technology and one of the pioneer science and technology that combines the systems theory, control theory and operation research.

Mathematically, HDSs refer to those dynamic systems whose state space consists of both the Euclidean space and the bounded set of the discrete events. In such systems, the evolution of the systems states is driven by both the continuous time and the discrete event. Markovian jump systems are one important class of HDSs, where the discrete events are governed by a Markov process. Various successful applications

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of MJSs can be seen in modern communication technology, fault tolerance control, etc., and the related theoretical challenges have attracted interests over the last several decades.

Since introduced by Krasovskii Lidskii in 1961, the research on MJSs has attracted much attention from almost all aspects of the control community. Stability has always been an important focus in MJSs [12, 27, 84]. For example, Mariton obtained the sufficient conditions for the mean-square stability and stability in probability of MJSs using the Lyapunov method, and then the sufficient and necessary conditions using Kronecker product [67–69]; Feng et.al. proved the equivalence between the second order moments of MJSs which is sufficient for the stability in probability [36]; Other works on the stability of MJSs can be seen in, e.g., [1, 11, 18, 20, 26, 32, 35, 36, 49, 50, 75, 81]. Based on those studies, other properties like controllability [49, 50, 88], observability [24, 25, 49, 50, 92], optimal control [22, 23, 33, 37, 39, 49, 50], and so on, have also been studied. Since 1990 s the theoretical foundations for linear MJSs in continuous time have been constructed. Ongoing are more general forms of MJSs like MJSs in discrete time [20, 23, 36, 88], nonlinear MJSs [85, 96, 100–102], and so forth.

Since a large number of practical systems can be modelled as MJSs, the study of MJSs then has both theoretical as well as practical importance. We also notice that significant improvements on the theory of MJSs are still needed. This book will cover a wide range of topics related to MJSs, e.g., the robustness and the practical stability of MJSs, its applications in mobile robots and networked control systems, etc. These discussions should be of interest to the reader in this field.

1.1.1 Robust Stochastic Stability

Research on this area has been reported extensively, see, e.g., [9–11, 13–16, 19, 22]. To name a few, Boukas and Liu proposed a guaranteed cost robust control strategy for uncertain discrete time MJSs in [14], Chen and Benjelloun et al. provided solutions for uncertain MJSs with and without delays, respectively in [13, 22], and output feedback based strategies can be found in [16]. However those works are restrained to know bounded norms for the unknown parts [10], or some other forms of restrictions are enforced, and works are less seen for more general cases with unknown norm bound.

Lasalle stability principle is often the foundational basis of the parameter estimation based design of the adaptive control strategy for general non-jump systems, and therefore the key problem in dealing with jump systems is also the construction of the corresponding Lasalle stability principle. This has been the central of the research in the last several decades. For example, in 1990 Ji and Chizeck in [50] and Mariton in [70] discussed the asymptotic stability of linear jump systems

$$\dot{x}(t) = A(r(t))x(t) + B(r(t))u(t).$$

In 1996 Shaikhet included delay in the discussion, that is,

$$dx(t) = A_1(r(t))x(t)dt + A_2(r(t))x(t-\tau)dt + \sigma(x(t), r(t))dB(t).$$

Mao proposed such stability conditions for more general nonlinear and delayed jump systems in [64, 66]. In all these works, the objective is to make the system state approach to zero (in probability or in mean-square).

Mao [65] and Deng [30] have set up the Lasalle stabilility principle for general nonlinear stochastic systems by different means,

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t).$$

In 1996 Basak proposed the local asymptotic stability concept for semi-linear jump systems [7]

$$dx(t) = A(r(t))x(t)dt + \sigma(x(t), r(t))dB(t),$$

and then Mao extended this to nonlinear systems

$$dx(t) = f(x(t), r(t))dt + \sigma(x(t), r(t))dB(t).$$

All these works lay the foundation of the further progress on the theory of jump systems.

1.1.2 Imprecise Jumping Parameters

The main theoretical foundation for the robust analysis and synthesis of systems with uncertain parameters in the time domain is the Lyapunov stability theory. In the early days one main method is to use the Riccati equation, which convents the problem to the solvability of a Riccati type matrix equation, and then gives the conditions for robust stability as well as the design method for the robust controller. This method pre-requires certain parameters to be given, and the selection of the parameters has a significant effect on the solvability and then the conservativeness. Since 1990s, linear matrix inequality (LMI) based methods become popular, since many control problems can be converted to the feasibility of a set of LMIs, or the convex optimization problem subject to some LMI-based constraints. This convex constraint means that a set of controllers can be obtained subject to the predetermined constraints, which is particularly useful in dealing with multi-target control problems. The `LMI toolbox` developed by `MATLAB` provides us with the powerful computational tool for LMI based design.

On the other hand, H_∞ control has been a fast developing field in control theory since its first introduction by Zames in 1981 [103]. Related works include, e.g., Youla based parametric, Nevanlinna-Pick theory and model fitting method, the solution to

Riccati equation in the time domain, and so on. Furthermore, the robust control toolbox in MATLAB makes H_∞ control theory a practical solution to engineering systems [31, 86, 99]. Works on the robust control of linear jump systems based on H_∞ theory have also been reported [2, 8, 21, 87, 90, 97].

One important assumption in exiting works is that the mode of the Markov jump parameter $r(t)$ is accurately measurable, which, however, is often impossible, due to the poor quality of the device or external disturbance. Therefore, it becomes more and more important to design the robust controller in the presence of inaccurate measurement of Markov jump parameters.

1.1.3 Nonlinear Markovian Jump Systems

Practical systems are essentially nonlinear. The nonlinearity can be intrinsic to the control system, or due to the practical constraints such as the saturation, or created by the nonlinear control law like the Bang-bang control. Works have been reported for nonlinear jump systems in recent years. These works have considered nonlinear jump systems with uncertain parameters [2, 17, 76], the stability of such systems [3, 4], filter design [94], jump parameter detection and filter design [71], robust control in the presence of the Lure term [77], and so on.

One common assumption in these existing works is that the the nonlinear terms are known or upper bounded by a known bound [17, 77], and these bounds are often needed in the controller design. These assumptions may not be feasible in practice. For such cases adaptive control may be useful. Works have been done for the robust adaptive control for deterministic nonlinear jump systems, including the work proposed by [43, 80].

On the other hand, the global stabilization of nonlinear systems has been a pioneering field in control theory. The Lyapunov theory is one of the main basis for such systems [6, 89]. It is noticed that no universal methodology exists for all nonlinear systems, but for those with strict feedback form or equivalent nonlinear systems, backstepping method is probably the most efficient solution [54]. A large volume of results in recent years have proven the effectiveness of the backstepping method [34, 40–42, 51–53, 78, 79]. Some other works can also be seen in [38] for the inverse optimization method, and the extensions of the backstepping method in various cases [5, 28–30, 58–60].

Though effective, no results on the controller design for nonlinear MJSs based on the backstepping method have been reported to date. It is known that the first step of the backstepping method is the state transformation, and then the construct of the Lyapunov function and virtual controller based on the new state. This can be fine for general continuous nonlinear systems, but the Markov jump parameters make that the transformed states are dependent on those parameters and are not continuous. This difficulty proposes great challenges for the controller design and stability analysis for nonlinear MJSs.

1.1.4 Practical Stability

One essential problem in studying the mathematical models for various practical systems is the stability. Lyapunov stability was first proposed in 1892 by Lyapunov in his PhD thesis. In such a theory the properties of the solutions to a set of n-dimensional differential equations are converted to the discussion of a scale function (the so-called Lyapunov function) and its derivatives, successfully constructing the fundamental framework of general stability theory. This theory and associated tools have been widely applied to various areas, including both deterministic systems [47, 57, 98] and stochastic ones [44, 62, 63].

Another stability definition is practical stability. From the practical viewpoint, a system can be thought of as stable if its solution is within certain region around the equilibrium. This is not mathematically stable but often acceptable in practice. For example, a rocket may contain trajectories which are unstable in the mathematical sense but can be practically acceptable. This fact thus derives another stability definition, i.e., the so-called practical stability, which was first introduced by LaSalle and Lefschetz in 1961 [56], and further improved later on by Lakshmikantham [55], Martynyuk [72, 73], and so on. The general theory for practical stability is still ongoing. In this book we will discuss the practical stability of MJSs in the probability sense and the corresponding controllability and optimal control problems.

1.1.5 Networked Control Systems

Networked control systems (NCSs) are control systems that are closed via some form of communication networks [45]. These communication networks can be either control-oriented, such as the Control Area Network (CAN), DeviceNet, etc., or non-control-optimized, like the widely used Internet. Most challenges emerge due to the introduction of the communication networks to the control systems, since lossless and real-time data transmission are usually not guaranteed by the communication network, especially those data networks that are not specifically optimized for the real-time control purpose [46, 74, 82, 83, 93, 95, 104].

MJSs can play a significant role in the development of NCSs, since NCSs are essentially composed of two different types of signals, i.e. the controlled plant which is usually in continuous time, and the computer-based data transmission which is essentially in discrete time, thus making “hybrid” and “switch” some intrinsic features of NCSs. Considerable works have been reported on the MJS modelling and analysis of NCSs, and more works are still expected for the future development of NCSs.

1.2 Model Description and Preliminaries

As pointed out earlier, the state space of Hybrid Dynamic Systems consists of both the Euclidean vector space $\mathbb{R}^n$ and the set of the discrete events $\mathscr{S}$, and can be categorized as two types, those in discrete time and in continuous time, respectively. We now give the basic model for HDSs in continuous time with Markov jump parameters [70].

Let $r(t)$ be a Markov chain in continuous time defined on the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ with its domain being $\mathscr{S} = \{1, 2, \cdots, N\}$. Each $r(t) \in \mathscr{S}$ is called a "regime" of the system. Ω is the sample space, $\mathscr{F}$ is σ-algebra, $\{\mathscr{F}_t\}_{t\geq 0}$ is the sub-σ-algebra reference set which is continuous from the right on t, $\mathscr{F}_{t_1} \subset \mathscr{F}_{t_2} \subset \mathscr{F}$, $(t_1 < t_2)$, $\mathscr{F}_0$ contains all P-null set, and P is the probability measure. Then the basic model for a Markovian jump system can be described as follows,

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), r(t), t), \\ y(t) = h(x(t), r(t), t), \end{cases} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the Euclidean vector space, representing the state, input and output, respectively, $f(\cdot), h(\cdot)$ are, respectively, the analytic mapping of $\mathbb{R}^n \times \mathbb{R}^m \times \mathscr{S} \times \mathbb{R} \to \mathbb{R}^n$ and $\mathbb{R}^n \times \mathscr{S} \times \mathbb{R} \to \mathbb{R}^p$ which satisfy the general increase and smooth conditions [70], to ensure the unique solution for $x(t_0)$ and $u(t)$ under arbitrary regime and initial state.

Let $\phi_t \in \mathbb{R}^N$ be the characteristic function of $r(t)$, i.e.,

$$\phi_{ti} = \begin{cases} 1, & r(t) = i; \\ 0, & r(t) \neq i, \end{cases} \quad i = 1, 2, \cdots, N.$$

Then the functions in the jump system can be described by the following three expressions,

$$f(\cdot, r(t)), \quad f(\cdot, \phi_t), \quad \sum_{i=1}^{N} f_i(\cdot)\phi_{ti},$$

and ϕ_t satisfies

$$d\phi_t = \Pi'\phi_t dt + dM_t, \tag{1.2}$$

where M_t is $\{\mathscr{F}_t\}$-martingale, $\Pi = [\pi_{ij}]_{i,j\in\mathscr{S}}$ is the state transition matrix of $r(t)$ given by

$$P\{r(t+\Delta) = j|r(t) = i\} = \begin{cases} \pi_{ij}\Delta + o(\Delta), & i \neq j; \\ 1 + \pi_{ii}\Delta + o(\Delta), & i = j, \end{cases} \tag{1.3}$$

where

$$\lim_{\Delta \to 0} o(\Delta)/\Delta = 0, \qquad (\Delta > 0),$$

and

$$\pi_{ii} = -\sum_{j \neq i} \pi_{ij}, \quad (\pi_{ij} \geq 0, j \neq i). \tag{1.4}$$

For simplicity hereafter we assume $t_0 = 0, x(0) = x_0, r(0) = r_0$ are constants. For matrix $F(r(t))$, we may simplify $F(r(t))$ as F_i for $r(t) = i$, i.e., $F_i \triangleq F(r(t))|_{r(t)=i}$, when no confusion is caused.

Remark 1.1 Another category of HDSs is the switched systems described by

$$\dot{x}(t) = f_{\sigma(t,x(t))}(t, x(t), u(t)), \quad x(t_0) = x_0, \tag{1.5}$$

where $x(\cdot) \in \mathbb{R}^n$ is the system state, $u(\cdot) \in \mathbb{R}^m$ is the input, $\sigma(t, x) : [t_0, +\infty) \times \mathbb{R}^n \to \mathscr{I}$ ($\mathscr{I}$ is the index set which can be infinity) is piecewise constant on (t, x) and right continuous on t, referred to as the switch law or switch strategy for system (1.5). The switch time instant is

$$t_k = \inf \left\{ t > t_{k-1} : \sigma(t, x(t)) \neq \sigma(t_{k-1}, x(t_{k-1})) \right\}, \quad k = 1, 2, \cdots$$

where

$$\inf_{k \geq 0} (t_k - t_{k-1}) = \limsup_{t \to \infty} \frac{\text{Number of switches in } [t_0, t)}{t - t_0}$$

are the dwell time and switch frequency of $\sigma(\cdot, \cdot)$, respectively. The switch law $\sigma(t, x)$ can be dependent on the events defined by the time and system state. Further, the switch law is controlled if $\sigma(t, x)$ is dependent on u as well [91].

The main difference between the jump system studied in this book and general swithed systems is that the discrete dynamics in the former is uncontrolled and independent on the system state, while switch itself can be a way of stabilization for the latter.

The following definitions and theories are needed.

Definition 1.1 (*Infinitesimal operator*) The effect of the infinitesimal operator of $(x(t), r(t))$, $\mathfrak{L}$, on scale function $g(x(t), r(t), t)$, is defined as

$$\begin{aligned}&\mathfrak{L}g(x(t), r(t), t)\\&= \lim_{\Delta\to 0} \frac{1}{\Delta} \left[E\left\{g(x(t), r(t), t) \left|x(t^-), r(t^-), t^-\right.\right\} - g(x(t^-), r(t^-), t^-)\right],\end{aligned}$$

where $\Delta = t - t^-$.

Definition 1.2 (*Uniform boundness in probablity*ρ) The state x_t of a HDS is uniformly bounded in probability ρ with the bound ε if the solution of the HDS $x_t(x_0)$ is such that

$$P\left\{\sup_{0\le t<\infty} ||x_t(x_0)|| \ge \varepsilon\right\} \le 1-\rho. \tag{1.6}$$

Definition 1.3 (*Uniform boundness with probablity 1*) The state x_t of a HDS is uniformly bounded with probability 1 and the bound is ε if the solution of the HDS $x_s(x_0)$ is such that

$$\lim_{t\to\infty} P\left\{\sup_{s\ge t} ||x_s(x_0)|| \ge \varepsilon\right\} = 0. \tag{1.7}$$

Definition 1.4 (*Stochastic stability*) A HDS is stochastically stable if the solution of the HDS $x_t(x_0)$ is such that

$$\int_0^\infty E\left\{\|x_t(x_0)\|^2\right\} dt < \infty.$$

Definition 1.5 (*Stability with probability 1*) The equilibrium of a HDS is stable with probability 1 if the solution of the HDS $x_t(x_0)$ is such that

$$P\left\{\lim_{||x_0||\to 0} \sup_{0\le t<\infty} ||x_t(x_0)|| = 0\right\} = 1. \tag{1.8}$$

Definition 1.6 (*Asymptotic stability in mean-square*) A HDS is asymptotically stable in mean-square if the solution of the HDSs $x_t(x_0)$ is such that

$$\lim_{t\to\infty} E\left\{\|x_t(x_0)\|^2\right\} = 0.$$

Definition 1.7 (*Asymptotic stability with probability 1*) The equilibrium of a HDS is asymptotically stable with probability 1 if for any $\varepsilon > 0$, there exists $\delta > 0$, such that when $\| x_0 \| \le \delta$ the solution of the HDS $x_t(x_0)$ satisfies

$$\lim_{t\to\infty} P\left\{\sup_{s\ge t} \| x_s(x_0) \| \ge \varepsilon\right\} = 0. \tag{1.9}$$

Definition 1.8 (*Exponential stability in p-th moment*) A HDS is exponentially stable in p-th moment if there exists $\alpha > 0, \beta > 0$ such that its solution $x_t(x_0)$ satisfies

$$E\{||x_t(x_0)||^p\} \leq \beta ||x_0||^p e^{-\alpha t}.$$

In particular, it is exponentially stable in mean-squareif $p = 2$.

Consider the following switched stochastic nonlinear retarded systems

$$\mathrm{d}x = f(t, x_t, \nu, \sigma)\mathrm{d}t + g(t, x_t, \nu, \sigma)\mathrm{d}B, \tag{1.10}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $\nu(t) \in \mathscr{L}_\infty^l$ is the control input, $B(t)$ is the m-dimensional Brownian motion which is defined on the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geq t_0}, P)$, with Ω being a sample space.

Definition 1.9 [61](*stochastic input-to-state stability, SISS*) The system (1.10) is stochastic input-to-state stability(SISS) if for any given $\varepsilon > 0$, there exist a $\mathscr{KL}$function $\beta(\cdot, \cdot)$, a $\mathscr{K}$function $\gamma(\cdot)$ such that

$$P\{|x(t)| < \beta(|x_0|, t) + \gamma(\|u\|_{[0,t)})\} \geq 1 - \varepsilon, \qquad \forall\, t \geq 0, \forall\, x_0 \in \mathbb{R}^n \tag{1.11}$$

where $\|u(s)\| = \inf_{\mathscr{A} \subset \Omega, P(\mathscr{A})=0} \sup\{|u(\omega, s)| : \omega \in \Omega \backslash \mathscr{A}\}$, $\|u\|_{[0,t)} = \sup\limits_{s \in [0,t)} \|u(s)\|$.

Definition 1.10 [61](*globally asymptotically stability in probability, GASiP*) The equilibrium $x = 0$ of system (1.10) is globally asymptotically stable in probability (GASiP) if for any $\varepsilon > 0$ there exists $\mathscr{KL}$ function $\beta(\cdot, \cdot)$ such that, with the input $u = 0$,

$$\mathbb{P}\{|x(t)| < \beta(|x_0|, t - t_0)\} \geq 1 - \varepsilon, \quad \forall t \geq t_0. \tag{1.12}$$

Definition 1.11 [48](*p*th *moment, ISS*) The system (1.10) is said to be pth ($p > 0$) moment input-to-state stable if there exist $\beta \in \mathscr{KL}$ and $\gamma \in \mathscr{K}$ such that the solution $x(t) = x(t; t_0, x_0, i_0)$ satisfies

$$E|x(t)|^p \leq \beta(E|x_0|^p, t) + \gamma(\|u\|_\infty), \ \forall\, t \geq 0. \tag{1.13}$$

for any essentially bounded input $u \in \mathbb{R}^m$ and any initial data $x_0 \in \mathbb{R}^n, i_0 \in \mathscr{S}$, where $\|u\|_\infty = \sup_{s \in [0,\infty)} \|u(s)\|$.

Definition 1.12 [105] (*input-to-state stable in mean, ISSiM*) The system (1.10) is input-to-state stable in mean (ISSiM) if there exist $\beta \in \mathscr{KL}$ and $\alpha, \gamma \in \mathscr{K}_\infty$, such that for any $u \in \mathbb{R}^m$, $x_0 \in \mathbb{R}^n$, we have

$$E[\alpha(|x(t)|)] \leq \beta(|x_0|, t) + \gamma(\|u\|_{[0,t)}), \forall\, t \geq 0. \tag{1.14}$$

Theorem 1.1 *[70] Consider the following HDS,*

$$\dot{x}(t) = f(x(t), u(t), r(t), t),$$

where for any $r(t) \in \mathscr{S}$, $f(\cdot)$ are continuous on t, $x(t)$, and the increase and smooth conditions are satisfied so that the unique solution exists for any regime and initial state. Let $g(x(t), r(t), t)$ be a scalar function of x(t), r(t), and t. Then, for $r(t) = i$, the infinitesimal generator, $\mathfrak{L}$, is

$$\mathfrak{L}g(x(t), i, t) = g_t(x(t), i, t) + f^T(x(t), u(t), i, t)g_x(x(t), i, t) + \sum_{j=1}^{N} \pi_{ij} g(x(t), j, t), \tag{1.15}$$

where $g_t(x(t), i, t)$, $g_x(x(t), i, t)$ are the partial derivatives on t and $x(t)$, respectively.

Theorem 1.2 *[36] For linear HDSs*

$$\begin{cases} \dot{x}(t) = A(r(t))x(t), \\ x(0) = x_0, \end{cases} \tag{1.16}$$

the following statements hold,

a. Asymptotic stability in mean-square, expontial stability in mean-square and stochastic stability are equivalent.

b. Stability almost everywhere can be inferred from asymptotic stability in mean-square, expontial stability in mean-square or stochastic stability, but not true vice versa.

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Chapter 2
Robust Stochastic Stability

This chapter investigates the robust output feedback H_∞ control for a class of uncertain Markovian jump linear systems with mode-dependent time-varying time delays. With known bounds of the system uncertainties and the control gain variations, we develop the sufficient conditions to guarantee the robust stochastic stability and the γ-disturbance H_∞ attenuation for the closed-loop system. These conditions can be solved by `LMI Toolbox` efficiently. Note here that the control design is based on the measured Markovian jumping parameter r_t^o that may be inconsistent with the true jumping parameter r_t due to the measurement noises.

2.1 Introduction

Robust stability for time-delayed Markovian jump systems with uncertainties has always been a challenging problem and has been widely investigated so far. In this field H_∞ design has been one popular tool for uncertain delayed Markovian jump system due to its capability of dealing with disturbance attenuation [2, 7, 8]. For example, in [7], the results for the robust stochastic stability and γ-suboptimal H_∞ state-feedback controller design were presented. In [2], a sufficient condition for robust stochastic stability and H_∞-disturbance attenuation was derived for a class of uncertain delayed Markovian jump linear systems based on the Lyapunov functional method, where the uncertainties are of the norm-bounded type. In [8] the delay-dependent H_∞ control problem was considered by adopting a descriptor model transformation method and a new bounding inequality.

In most existing works, the jumping parameters are often assumed to be precisely known. This assumption is usually not true in practice while the system states can often be observed. Therefore, the Wonham filter can be used to estimate the jumping parameters using the given system matrices. To address this problem, the adaptive stabilization was studied in [6], where the existence condition and the adaptive certainty

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equivalence feedback control were proposed by the parameter estimation technique of nonlinear filters. On the other hand, imprecise measurements are often present in analog systems and quantization error sometimes can not be ignored in digital control systems, making precise control implementation almost impossible. To make it worse, the overall systems will have poor stability margins if these robust control strategies are not properly implemented, which applies to common techniques such as H_∞, l_1 or μ synthesis, etc.

On a parallel line, time delays often exist in practical systems such as mechanical systems, chemical processes, neural networks. Delays can deteriorate the system performance or even unstabilize the system. For the stability analysis and controller design of such delayed systems the Lyapunov-Krasovskii functionals (LKFs) approaches are widely used [14–16]. In order to reduce the conservatism caused by model transformations and inequalities, many new techniques were proposed for uncertain time delay systems [11, 12, 22, 23]. In [24], the free-weighting matrix method was proposed to bound the cross product terms and it can reduce the conservatism greatly.

In this chapter, we consider the problem of robust output-feedback H_∞ control for a class of uncertain Markovian jump linear systems with mode-dependent time-varying delays. We also consider the measurement errors of the jumping parameters, which are always inevitable due to the detection delays and false alarm of the identification algorithms [20]. The robust stochastic stability analysis and H_∞ disturbance attenuation design are given by using the measurement value of the jumping parameters directly.

2.2 Uncertain Markovian Jump Linear Systems with Time Delays

Consider the following uncertain Markovian jump linear stochastic systems with mode-dependent time-varying delays,

$$\begin{cases} \dot{x}(t) = [A_1(r_t) + \Delta_{A_1}(r_t, t)]x(t) + [A_2(r_t) + \Delta_{A_2}(r_t, t)]x(t - \tau_{r_t}(t)) \\ \qquad\quad +[B_1(r_t) + \Delta_{B_1}(r_t, t)]u(t) + B_2(r_t)w(t), \\ z(t) = [C(r_t) + \Delta_C(r_t, t)]x(t), \\ x(s) = f(s), \quad r_s = r_0, \quad s \in [-2\mu, \ 0], \end{cases} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^{m_3}$ $u(t) \in \mathbb{R}^{m_1}$ are the system states, system outputs, and control inputs, respectively. $A_1(r_t) \in R^{n\times n}$, $A_2(r_t) \in \mathbb{R}^{n\times n}$, $B_1(r_t) \in \mathbb{R}^{n\times m_1}$, $B_2(r_t) \in \mathbb{R}^{n\times m_2}$, $C(r_t) \in \mathbb{R}^{m_3\times n}$ are known real matrices denoting the nominal system parameters, and $\Delta_{A_1}(r_t, t) \in R^{n\times n}$, $\Delta_{A_2}(r_t, t) \in R^{n\times n}$, $\Delta_{B_1}(r_t, t) \in R^{n\times m_1}$, $\Delta_C(r_t, t) \in R^{m_3\times n}$ are unknown matrices representing the model uncertainties [2, 7, 9]. $w(t) \in \mathbb{R}^{m_2}$ is the exogenous disturbance input which satisfies $w(t) \in L_2[0, \infty)$. $f(t) \in \mathbb{R}^n$ is a continuous function denoting the initial states. r_t is a continuous-time

Markov chain that takes value in finite set $\mathscr{S} = \{1, 2, \ldots, N\}$ with the transition rate matrix Π defined in (1.3). $\tau_{r_t}(t)$ represents the mode-dependent time-varying delay that satisfies

$$0 < \tau_{r_t}(t) \leq \mu_{r_t} \leq \mu < \infty, \quad \dot{\tau}_{r_t}(t) \leq h_{r_t} < 1, \quad \forall\, r_t \in \mathscr{S} \tag{2.2}$$

where μ_{r_t} and h_{r_t} are upper bounds of $\tau_{r_t}(t)$ and $\dot{\tau}_{r_t}(t)$, for given $r_t \in \mathscr{S}$. μ is the common upper bounded and can be set as $\mu = \max_{i \in \mathscr{S}}\{\mu_i\}$.

The following assumption is necessary to establish the main results.

Assumption 2.1 The uncertain parameters can be written as follows [27]:

$$\begin{aligned}
\Delta_{A_1}(r_t, t) &= H_1(r_t)F(r_t, t)E_1(r_t),\\
\Delta_{A_2}(r_t, t) &= H_1(r_t)F(r_t, t)E_2(r_t),\\
\Delta_{B_1}(r_t, t) &= H_1(r_t)F(r_t, t)E_3(r_t),\\
\Delta_{C}(r_t, t) &= H_2(r_t)F(r_t, t)E_4(r_t),
\end{aligned}$$

where $H_1(r_t) \in \mathbb{R}^{n \times n_f}$, $H_2(r_t) \in \mathbb{R}^{m_3 \times n_f}$, $E_1(r_t) \in \mathbb{R}^{n_f \times n}$, $E_2(r_t) \in \mathbb{R}^{n_f \times n}$, $E_3(r_t) \in \mathbb{R}^{n_f \times m_1}$ and $E_4(r_t) \in \mathbb{R}^{n_f \times n}$ are known real matrices, while $F(r_t, t) \in \mathbb{R}^{n_f \times n_f}$ are the uncertain matrix functions satisfying

$$F^T(r_t, t)F(r_t, t) \leq I, \quad \forall\, r_t \in \mathscr{S}. \tag{2.3}$$

Remark 2.1 As an extension of the **matching condition**, the structure of the uncertainties in Assumption 2.1 is widely used in the literature on robust control and robust filter, see e.g. [1–4, 7–9, 27]. How the uncertain matrix functions $F(r_t, t)$ affect the nominal parameters $A_1(r_t)$, $A_2(r_t)$, $B_1(r_t)$, $C(r_t)$ can be characterized by $H_1(r_t)$, $H_2(r_t)$, $E_1(r_t)$, $E_2(r_t)$, $E_3(r_t)$ and $E_4(r_t)$

In practical control systems, the environmental noises, external disturbance and other modelling uncertainties unavoidably cause detection delays and false alarms when we identify the activated system mode. Similar to [19, 20], we adopt two stochastic processes to describe the above phenomena. One process, denoted by r_t, is used to characterize the actual system mode in (2.1), and the other one, denoted by r_t^o, represents the mode we observed or measured in the practical systems. The difference between r_t and r_t^o are mainly caused by two kinds of measurement errors, i.e. the detection delays and false alarms. The following models are used to describe these measurement errors.

- The probability of jump from i to j conditional on r_t, denoted by r_t^o, can be written as

$$P\left\{r_{t+\Delta}^o = j \left| \begin{array}{l} r_s^o = i \\ r_{t_0} = j \\ r_{t_{0-}} = i \\ s \in [t_0,\ t] \end{array} \right.\right\} = \begin{cases} \pi_{ij}^0 \Delta + o(\Delta), & i \neq j \\ 1 + \pi_{ii}^0 \Delta + o(\Delta), & i = j \end{cases} \tag{2.4}$$

In fact, r_t^o can be seen as an exponentially distributed random variable with rate π_{ij}^o. The parameters π_{ij}^o can be obtained by evaluating observed sample paths, and

$$\pi_{ii}^0 = -\sum_{j \neq i} \pi_{ij}^0, \quad (\pi_{ij}^0 \geq 0,\ j \neq i). \tag{2.5}$$

- Although r_t remains at i, r_t^o can still occasionally transmit from i to j. Similarly, we also use an independent exponential distribution with mean $1/\pi_{ij}^1$ to describe this scenario

$$P\left\{r_{t+\Delta}^o = j \left| \begin{array}{l} r_s^o = i \\ s \in [t_0,\ t] \end{array} \right.\right\} = \begin{cases} \pi_{ij}^1 \Delta + o(\Delta), & i \neq j \\ 1 + \pi_{ii}^1 \Delta + o(\Delta), & i = j \end{cases} \tag{2.6}$$

where π_{ij}^1 is the false alarm rate, which can also be evaluated from observed sample paths, and satisfies

$$\pi_{ii}^1 = -\sum_{j \neq i} \pi_{ij}^1, \quad (\pi_{ij}^1 \geq 0,\ j \neq i). \tag{2.7}$$

For simplicity, we simplify $M(r_t^o, r_t, t)$ as $M_{ji}(t)$ when $r_t^o = j, r_t = i,\ j, i \in \mathscr{S}$, and let the initial time $t_0 = 0$, then the initial conditions can be written as $x(0) = x_0, r_0$ and r_0^o. Note that all these initial value are deterministic.

The following dynamic output feedback controllers are to be designed.

$$\begin{cases} \dot{\widehat{x}}(t) = A_3(r_t^o)\widehat{x}(t) + B_3(r_t^o)z(t), \\ u(t) = K(r_t^o)\widehat{x}(t), \end{cases} \tag{2.8}$$

where $\widehat{x}(t) \in \mathbb{R}^n$ is the states of the controllers, and $A_3(r_t^o)$, $B_3(r_t^o)$, $K(r_t^o)$ are the unknown matrices of the controllers with appropriate dimensions to be determined.

Practically, it is impossible to implement the above controllers precisely. So, in this chapter, the controllers with imprecise implementation are described as

$$u(t) = [I + \alpha(r_t)\phi(r_t, t)]K(r_t^o)\widehat{x}(t), \tag{2.9}$$

where $\alpha(r_t)\phi(r_t, t)$ represent the additive errors that affect the controller gains. $\alpha(r_t)$ is a positive constant and $\phi(r_t, t)$ satisfies

$$\phi^T(r_t, t)\phi(r_t, t) \leq I, \quad \forall r_t \in \mathscr{S}.$$

Remark 2.2 Notice that the designed controllers are dependent on the measured jumping parameter r_t^o. To reconfigure the controllers, the switching of controller gains $K(r_t^o)$ is based on r_t^o. However, the evolution of the dynamic systems follows the actual mode r_t, and therefore, the variations of the controller gains depend on r_t and have nothing to do with r_t^o.

Apply the control law (2.8) to system (2.1) and denote $\xi(t) = [x^T(t), \widehat{x}^T(t)]^T$, we obtain the closed-loop system

$$\begin{cases} \dot{\xi}(t) = \overline{A}_1(r_t^o, r_t, t)\xi(t) + \overline{A}_2(r_t)I_0\xi(t - \tau_{r_t}(t)) + \overline{B}_2(r_t)w(t), \\ z(t) = [C(r_t) + \varDelta_C(r_t, t)]I_0\xi(t), \\ I_0\xi(s) = f(s), \quad r_s = r_0, \quad s \in [-2\mu, \ 0], \end{cases} \tag{2.10}$$

where

$$\overline{A}_{1ji} = \begin{bmatrix} A_{1i} + \varDelta_{A_{1i}}(t) & \left(B_{1i} + \varDelta_{B_{1i}}(t)\right)(I + \alpha_i\phi_i(t))\,K_j \\ B_{3j}\left(C_i + \varDelta_{C_i}(t)\right) & A_{3j} \end{bmatrix} \in \mathbb{R}^{2n\times 2n},$$

$$\overline{A}_{2i} = \begin{bmatrix} A_{2i} + \varDelta_{A_{2i}}(t) \\ 0 \end{bmatrix} \in \mathbb{R}^{2n\times n}, \quad \overline{B}_{2i} = \begin{bmatrix} B_{2i} \\ 0 \end{bmatrix} \in \mathbb{R}^{2n\times m_2},$$

$$I_0 = \begin{bmatrix} I & 0 \end{bmatrix} \in \mathbb{R}^{n\times 2n}, \, for\ each\ r_t^o = j, r_t = i, \quad \forall i, j \in \mathscr{S}.$$

The objectives of this chapter are as follows:

(i) *Robust stabilization*: Determine the nominal controller gains $K(r_t^o)$ in (2.9) and establish sufficient conditions for the system (2.1) such that the overall closed-loop system (2.10) is robustly exponentially stable in the mean square sense;

(ii) H_∞ *control problem*: Given a constant scalar $\gamma > 0$, determine the nominal control gain $K(r_t^o)$ in (2.9) and establish the sufficient conditions such that the resulting closed-loop system (2.10) is robustly stochastically stable with disturbance attenuation level γ under zero initial condition ($x(0) = 0$), that is

$$J = E\left\{ \int_0^T \left[z^T(t)z(t) - \gamma^2 w^T(t)w(t) \right] dt \right\} < 0, \forall\, w(t) \neq 0, w(t) \in \mathscr{L}_2[0, \infty). \tag{2.11}$$

2.3 Robust Control

In this section, we study the exponential mean-square stability of the time-delayed uncertain Markovian jump linear system (2.10) with $w(t) = 0$. The following lemmas are needed in deriving the stability conditions.

Lemma 2.1 *[17] Schur complement: Consider the following matrix of appropriate dimension*

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}, \qquad Q_{22} > 0, \tag{2.12}$$

then Q is positive definite if and only if $Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T > 0$.

Lemma 2.2 *[25] Given matrices $Q = Q^T$, H, E and $R = R^T > 0$ of appropriate dimensions, then*

$$Q + HFE + E^T F^T H^T < 0$$

for all F satisfying $F^T F \leq R$, if and only if there exits some $\rho > 0$ such that

$$Q + \rho H H^T + \rho^{-1} E^T R E < 0.$$

Lemma 2.3 *[19] The infinitesimal generator $\mathfrak{L}$ of random processes can be defined as follows.*

For the following jump systems

$$\dot{x}(t) = f(x(t), u(t), r_t^o, r_t, t),$$

suppose that $f(\cdot)$ is continuous for all its variables within their domain of definition, and satisfies the usual growth and smoothness hypothesis, $g(x(t), r_t^o, r_t, t)$ is a scalar continuous function of t and $x(t)$, $\forall r_t^o, r_t \in \mathscr{S}$. Then, the infinitesimal generator $\mathfrak{L}$ of the random process $\{x(t), r_t^o, r_t, t\}$ can be described as follows:

- *For $r_t^o = r_t = i$, we have*

$$\begin{aligned} &\mathfrak{L}g(x(t), i, i, t) \\ &= \lim_{\triangle \to 0} \frac{1}{\triangle} \left[E\left\{ g(x(t+\triangle), r_{t+\triangle}^o, r_{t+\triangle}, t+\triangle) | x(t) = x, r_t^o = i, r_t = i, t \right\} \right. \\ &\quad \left. - g(x, i, i, t) \right] \\ &= g_t(x, i, i, t) + f^T(x, u(t), i, i, t) g_x(x, i, i, t) + \sum_{j=1}^N \pi_{ij} g(x, i, j, t) \\ &\quad + \sum_{j=1}^N \pi_{ij}^1 g(x, j, i, t). \end{aligned} \tag{2.13}$$

- *For $r_t^o = j \neq r_t = i$, we have*

$$
\begin{aligned}
&\mathfrak{L}g(x(t), j, i, t)\\
&= \lim_{\triangle \to 0} \frac{1}{\triangle} \left[E\left\{ g(x(t+\triangle), r_{t+\triangle}^o, r_{t+\triangle}, t+\triangle) | x(t) = x, r_t^o = j, r_t = i, t \right\} \right.\\
&\quad \left. -g(x, j, i, t) \right]\\
&= g_t(x, j, i, t) + f^T(x, u(t), j, i, t) g_x(x, j, i, t) \\
&\quad + \pi_{ji}^0 g(x, i, i, t) - \pi_{ji}^0 g(x, j, i, t).
\end{aligned} \tag{2.14}
$$

Theorem 2.1 *Consider the uncertain delayed Markovian jump linear system with $w(t) = 0$. If there exist symmetric positive-definite matrices P_{ij}, Q, Z, positive semi-definite matrices X_{ji}, real matrices K_j, Y_{ji}, T_{ji} that are of appropriate dimensions and positive constants ρ_{1ji}, ρ_{2ji}, ρ_{3ji} such that*

$$
\overline{\mathbf{W}}_{ji} = \begin{bmatrix}
L_1 & P_{1ji}B_{1i} + \rho_{3ji}E_{1i}^T E_{3i} & 0 & P_{1ji}H_{1i} & 0 & K_j^T \\
L_2 & 0 & K_j^T & 0 & V_{ji}^T H_{2i} & 0 \\
L_3 & \rho_{3ji}E_{2i}^T E_{3i} & 0 & 0 & 0 & 0 \\
L_4 & \mu Z B_{1i} & 0 & \mu Z H_{1i} & 0 & 0 \\
L_5 & -I + \rho_{3ji}E_{3i}^T E_{3i} & 0 & 0 & 0 & 0 \\
L_6 & 0 & -I + \rho_{1ji}\alpha_i^2 I & 0 & 0 & 0 \\
L_7 & 0 & 0 & -\rho_{3ji}I & 0 & 0 \\
L_8 & 0 & 0 & 0 & -\rho_{2ji}I & 0 \\
L_9 & 0 & 0 & 0 & 0 & -\rho_{1ji}I
\end{bmatrix} < 0 \tag{2.15}
$$

$$
\Gamma_{ji} = \begin{bmatrix}
X_{1ji} & X_{2ji} & I_0^T Y_{ji} \\
X_{2ji}^T & X_{3ji} & T_{ji} \\
Y_{ji}^T I_0 & T_{ji}^T & Z
\end{bmatrix} \geq 0, \quad \forall\, i, j \in \mathscr{S}. \tag{2.16}
$$

where

$$
\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \\ L_6 \\ L_7 \\ L_8 \\ L_9 \end{bmatrix} = \begin{bmatrix}
\Phi_{11} + \rho_{3ji}E_{1i}^T E_{1i} + \rho_{2ji}E_{4i}^T E_{4i} & C_i^T V_{ji} + \mu X_{1ji}^2 & \Phi_{13} + \rho_{3ji}E_{1i}^T E_{2i} & \mu A_{1i}^T Z \\
V_{ji}^T C_i + \mu X_{1ji}^{2^T} & U_{ji} + U_{ji}^T + \Phi_{22} & \mu X_{2ji}^2 & 0 \\
\Phi_{13}^T + \rho_{3ji}E_{2i}^T E_{1i} & \mu X_{2ji}^{2^T} & \Phi_{33} + \rho_{3ji}E_{2i}^T E_{2i} & \mu A_{2i}^T Z \\
\mu Z A_{1i} & 0 & \mu Z A_{2i} & -\mu Z \\
B_{1i}^T P_{1ji} + \rho_{3ji}E_{3i}^T E_{1i} & 0 & \rho_{3ji}E_{3i}^T E_{2i} & \mu B_{1i}^T Z \\
0 & K_j & 0 & 0 \\
H_{1i}^T P_{1ji} & 0 & 0 & \mu H_{1i}^T Z \\
0 & H_{2i}^T V_{ji} & 0 & 0 \\
K_j & 0 & 0 & 0
\end{bmatrix},
$$

$$X_{ji} = \begin{bmatrix} X_{1ji} & X_{2ji} \\ X_{2ji}^T & X_{3ji} \end{bmatrix} = \begin{bmatrix} X_{1ji}^1 & X_{1ji}^2 & X_{2ji}^1 \\ {X^2}_{1ji}^T & X_{1ji}^3 & X_{2ji}^2 \\ {X^1}_{2ji}^T & {X^2}_{2ji}^T & X_{3ji} \end{bmatrix}$$

with

$$\Phi_{11} = \begin{cases} \text{if } \ j = i \\ A_{1i}^T P_{1ii} + P_{1ii} A_{1i} + \sum_{j=1}^{N} \pi_{ij} P_{1ij} + \sum_{j=1}^{N} \pi_{ij}^1 P_{1ji} + Y_{ii} + Y_{ii}^T + (1+\eta\mu) Q + \mu X_{1ii}^1 \\ \text{if } \ j \neq i \\ A_{1i}^T P_{1ji} + P_{1ji} A_{1i} + \pi_{ji}^0 (P_{1ii} - P_{1ji}) + Y_{ji} + Y_{ji}^T + (1+\eta\mu) Q + \mu X_{1ji}^1 \end{cases}$$

$$\Phi_{22} = \begin{cases} \sum_{j=1}^{N} \pi_{ij} P_{2ij} + \sum_{j=1}^{N} \pi_{ij}^1 P_{2ji} + \mu X_{1ii}^3, & \text{if } \ j = i \\ \pi_{ji}^0 (P_{2ii} - P_{2ji}) + \mu X_{1ji}^3, & \text{if } \ j \neq i \end{cases}$$

$$\Phi_{13} = P_{1ji} A_{2i} - Y_{ji} + T_{ji}^T + \mu X_{2ji}^1, \quad \Phi_{33} = -T_{ji} - T_{ji}^T - (1-h_i) Q + \mu X_{3ji},$$

$$\eta = \max_{i \in \mathscr{S}} \{|\pi_{ii}|\}, \quad V_{ji} = B_{3j}^T P_{2ji}, \quad U_{ji} = A_{3j}^T P_{2ji},$$

then the systems (2.10) are exponentially stable in the mean-square sense.

Proof Consider the nominal time-delayed jump linear system Σ_0 without disturbance:

$$\Sigma_0 : \quad \begin{cases} \dot{\xi}(t) = \widehat{A}_1(r_t^o, r_t)\xi(t) + \widehat{A}_2(r_t) I_0 \xi(t - \tau_{r_t}(t)), \\ z(t) = [C(r_t) + \Delta_C(r_t, t)] I_0 \xi(t), \\ I_0 \xi(s) = f(s), \quad r_s = r_0, \quad s \in [-2\mu, \ 0] \end{cases} \tag{2.17}$$

where

$$\widehat{A}_1(r_t^o, r_t) = \begin{bmatrix} A_1(r_t) & B_1(r_t) K(r_t^o) \\ B_3(r_t^o) C(r_t) & A_3(r_t^o) \end{bmatrix} \in \mathbb{R}^{2n \times 2n},$$
$$\widehat{A}_2(r_t) = \begin{bmatrix} A_2(r_t) \\ 0 \end{bmatrix} \in \mathbb{R}^{2n \times n}.$$

It is worth pointing out that $\{(\xi(t), r_t^o, r_t), \ t \geq 0\}$ is non-Markovian due to the time delay $\tau_{r_t}(t)$. However, if we define a process $\{(\xi_t, r_t^o, r_t), t \geq 0)\}$ that taking values in $\mathscr{C}_0$, where $\xi_t = \{\xi(\theta + t) \mid -2\mu \leq \theta \leq 0\}$, $\mathscr{C}_0 = \bigcup_{i,j \in \mathscr{S}} \mathscr{C}[-2\mu, 0] \times \{i, j\}$, and $\mathscr{C}[-2\mu, 0]$ denotes the space of continuous functions on interval $[-2\mu, 0]$, then we can show that $\{(\xi_t, r_t^o, r_t), t \geq 0)\}$ is a strong Markov process with state space $\mathscr{C}_0$ [27].

Consider the following LKFs candidate:

$$V(\xi_t, r_t^o, r_t, t) = V_1 + V_2 + V_3 + V_4, \tag{2.18}$$

where

$$\begin{aligned}
V_1 &= \xi^T(t)P(r_t^o, r_t)\xi(t) = x^T(t)P_1(r_t^o, r_t)x(t) + \widehat{x}^T(t)P_2(r_t^o, r_t)\widehat{x}(t), \\
V_2 &= \int_{t-\tau_{r_t}(t)}^{t} x^T(s)Qx(s)ds \\
V_3 &= \eta \int_{-\mu}^{0}\int_{t+\theta}^{t} x^T(s)Qx(s)dsd\theta \\
V_4 &= \int_{-\mu}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z\dot{x}(s)dsd\theta.
\end{aligned}$$

For both cases of $r_t^o = r_t = i$ and $r_t^o = j, r_t = i, j \neq i$, we obtain their respective results according to the definition of the infinitesimal generator $\mathfrak{L}$ in Lemma 2.3.

Case I. $r_t^o = r_t = i$

We can find that

$$\begin{aligned}
\mathfrak{L}V_1 &= \xi^T(t)\left[\widehat{A}_{1ii}^T P_{ii} + P_{ii}\widehat{A}_{1ii} + \sum_{j=1}^{N}\pi_{ij}P_{ij} + \sum_{j=1}^{N}\pi_{ij}^1 P_{ji}\right]\xi(t) \\
&\quad + \xi^T(t)P_{ii}\widehat{A}_{2i}x(t-\tau_i(t)) + x^T(t-\tau_i(t))\widehat{A}_{2i}^T P_{ii}\xi(t), \\
\mathfrak{L}V_2 &= \xi^T(t)I_0^T QI_0\xi(t) - (1-\dot{\tau}_i(t))x^T(t-\tau_i(t))Qx(t-\tau_i(t)) \\
&\quad + \sum_{j=1}^{N}\pi_{ij}\int_{t-\tau_j(t)}^{t} x^T(s)Qx(s)ds \\
&\leq \xi^T(t)I_0^T QI_0\xi(t) - (1-h_i)x^T(t-\tau_i(t))Qx(t-\tau_i(t)) \\
&\quad + \sum_{j=1}^{N}\pi_{ij}\int_{t-\tau_j(t)}^{t} x^T(s)Qx(s)ds, \\
\mathfrak{L}V_3 &= \eta\mu\xi^T(t)I_0^T QI_0\xi(t) - \eta\int_{t-\mu}^{t} x^T(s)Qx(s)ds, \\
\mathfrak{L}V_4 &= \mu\dot{\xi}^T(t)I_0^T ZI_0\dot{\xi}(t) - \int_{t-\mu}^{t}\dot{x}^T(s)Z\dot{x}(s)ds \\
&\leq \mu\dot{\xi}^T(t)I_0^T ZI_0\dot{\xi}(t) - \int_{t-\tau_i(t)}^{t}\dot{x}^T(s)Z\dot{x}(s)ds.
\end{aligned}$$

Combining (1.4) and (2.2), we obtain

$$\sum_{j=1}^{N}\pi_{ij}\int_{t-\tau_j(t)}^{t}x^T(s)Qx(s)ds \le \sum_{j=1,j\neq i}^{N}\pi_{ij}\int_{t-\mu}^{t}x^T(s)Qx(s)ds$$
$$= -\pi_{ii}\int_{t-\mu}^{t}x^T(s)Qx(s)ds \le \eta\int_{t-\mu}^{t}x^T(s)Qx(s)ds. \qquad (2.19)$$

To overcome the conservativeness in selecting the optimal weighting matrices between the terms in the Newton-Leibniz formula, the following condition is presented [24]:

$$2\left[x^T(t)Y + x^T(t-d(t))T\right]\left[x(t) - \int_{t-d(t)}^{t}\dot{x}(s)ds - x(t-d(t))\right] = 0,$$

where the free weighting matrices Y and T indicate the relationship between the terms in the above formula, and they can easily be selected by means of linear matrix inequalities.

The following conditions are also employed to complete the proof.

$$\mu\zeta^T(t)X(r_t^o, r_t)\zeta(t) - \int_{t-\tau_{p_{mj,t}}(t)}^{t}\zeta^T(t)X(r_t^o, r_t)\zeta(t)ds \ge 0, \qquad (2.20)$$

$$2\left[\xi^T(t)I_0^T Y(r_t^o, r_t) + x^T(t-\tau_{r_t}(t))T(r_t^o, r_t)\right] \times$$
$$\left[I_0\xi(t) - \int_{t-\tau_{r_t}(t)}^{t}\dot{x}(s)ds - x(t-\tau_{r_t}(t))\right] = 0, \qquad (2.21)$$

where $\zeta^T(t) = [\xi^T(t) \;\; x^T(t-\tau_{r_t}(t))]$, and $X(r_t^o, r_t)$ are defined in Theorem 2.1.

We have

$$\mathfrak{L}V(\xi_t, i, i, t) \le \xi^T(t)\left[\widehat{A}_{1ii}^T P_{ii} + P_{ii}\widehat{A}_{1ii} + \sum_{j=1}^{N}\pi_{ij}P_{ij} + \sum_{j=1}^{N}\pi_{ij}^1 P_{ji}\right]\xi(t)$$
$$+ \xi^T(t)P_{ii}\widehat{A}_{2i}x(t-\tau_i(t)) + x^T(t-\tau_i(t))\widehat{A}_{2i}^T P_{ii}\xi(t)$$
$$+ (1+\eta\mu)\xi^T(t)I_0^T QI_0\xi(t) - (1-h_i)x^T(t-\tau_i(t))Qx(t-\tau_i(t))$$
$$+ \mu\dot{\xi}^T(t)I_0^T ZI_0\dot{\xi}(t) - \int_{t-\tau_i(t)}^{t}\dot{x}^T(s)Z\dot{x}(s)ds$$
$$+ 2\left[\xi^T(t)I_0^T Y(r_t^o, r_t) + x^T(t-\tau_{r_t}(t))T(r_t^o, r_t)\right]\Big[I_0\xi(t)$$

$$
\begin{aligned}
&- \int_{t-\tau_{r_t}(t)}^{t} \dot{x}(s)ds - x(t-\tau_{r_t}(t))\Big] + \mu\zeta^T(t)X(r_t^o, r_t)\zeta(t) \\
&- \int_{t-\tau_{r_t}(t)}^{t} \zeta^T(t)X(r_t^o, r_t)\zeta(t)ds \\
&= \zeta^T(t)\Xi_{ii}\zeta(t) - \int_{t-\tau_i(t)}^{t} \chi^T(t,s)\Gamma_{ii}\chi(t,s)ds,
\end{aligned} \tag{2.22}
$$

where

$$
\begin{aligned}
&\chi^T(t,s) = [\xi^T(t)\ \ x^T(t-\tau_i(t))\ \ \dot{x}^T(s)], \\
&\Xi_{ii} = \begin{bmatrix} \widehat{\Phi}_{11} + \mu\widehat{A}_{1ii}^T I_0^T Z I_0 \widehat{A}_{1ii} & \widehat{\Phi}_{12} + \mu\widehat{A}_{1ii}^T I_0^T Z I_0 \widehat{A}_{2i} \\ \widehat{\Phi}_{12}^T + \mu\widehat{A}_{2i}^T I_0^T Z I_0 \widehat{A}_{1ii} & \widehat{\Phi}_{22} + \widehat{A}_{2i}^T I_0^T Z I_0 \widehat{A}_{2i} \end{bmatrix}, \\
&\widehat{\Phi}_{11} = \begin{cases} \widehat{A}_{1ii}^T P_{ii} + P_{ii}\widehat{A}_{1ii} + \sum_{j=1}^{N}\pi_{ij}P_{ij} + \sum_{j=1}^{N}\pi_{ij}^1 P_{ji} + I_0^T Y_{ii} I_0 \\ \quad + I_0 Y_{ii}^T I_0^T + (1+\eta\mu)I_0^T Q I_0 + \mu X_{1ii}, & (\mathit{if}\ \ j = i) \\ \widehat{A}_{1ji}^T P_{ji} + P_{ji}\widehat{A}_{1ji} + q_{ji}^0(P_{ii} - P_{ji}) + I_0^T Y_{ji} I_0 + I_0 Y_{ji}^T I_0^T \\ \quad + (1+\eta\mu)I_0^T Q I_0 + \mu X_{1ji}, & (\mathit{if}\ \ j \neq i) \end{cases} \\
&\widehat{\Phi}_{12} = P_{ji}\widehat{A}_{2i} - I_0^T Y_{ji} + I_0^T T_{ji}^T + \mu X_{2ji}, \\
&\widehat{\Phi}_{22} = -T_{ji} - T_{ji}^T - (1-h_i)Q + \mu X_{3ji}, \\
&\eta = \max_{i\in\mathscr{S}}\{|\pi_{ii}|\}.
\end{aligned} \tag{2.23}
$$

If $\Xi_{ii} < 0$, $\Gamma_{ii} \geq 0$, then for each $i \in S$ and any scalar $\beta > 0$, we obtain

$$
\mathfrak{L}[e^{\beta t}V(\xi_t, i, i, t)] \leq -\alpha_1 e^{\beta t}\|\xi(t)\|^2 + \beta e^{\beta t}V(\xi_t, i, i, t), \ \ \forall\, i \in \mathscr{S}, \beta > 0, \tag{2.24}
$$

where $\alpha_1 = \min_{i\in\mathscr{S}}\{\lambda_{min}(-\Xi_{ii})\}$.

Similar to [27], we can verify that

$$
\begin{aligned}
V(\xi_t, i, i, t) &\leq \lambda_{max}(P_{ii})\|\xi(t)\|^2 + \lambda_{max}(Q)\int_{t-\tau_i(t)}^{t}\|x(s)\|^2 ds \\
&\quad + \eta\lambda_{max}(Q)\int_{-\mu}^{0}\int_{t+\theta}^{t}\|x(s)\|^2 ds d\theta + \lambda_{max}(Z)\int_{-\mu}^{0}\int_{t+\theta}^{t}\|\dot{x}(s)\|^2 ds d\theta \\
&\leq \lambda_{max}(P_{ii})\|\xi(t)\|^2 + (\eta\mu + 1)\lambda_{max}(Q)\int_{t-\mu}^{t}\|x(s)\|^2 ds \\
&\quad + \mu\lambda_{max}(Z)\int_{t-\mu}^{t}\|\dot{x}(s)\|^2 ds.
\end{aligned}
$$

Noticing that in nominal system Σ_0:

$$\dot{\xi}(t) = \widehat{A}_{1ii}\xi(t) + \widehat{A}_{2i} I_0 \xi(t - \tau_{r_t}(t))$$

and letting $\alpha_2 = \max_{i \in \mathscr{S}} \left\{ 2\|\widehat{A}_{1ii}\|^2 \right\}$, $\alpha_3 = \max_{i \in \mathscr{S}} \left\{ 2\|\widehat{A}_{2i}\|^2 \right\}$, it yields

$$\|\dot{\xi}(t)\|^2 \le \alpha_2 \|\xi(t)\|^2 + \alpha_3 \|x(t - \tau_{r_t}(t))\|^2.$$

This, together with (2.24), gives

$$\begin{aligned}
\mathfrak{L}[e^{\beta t} V(\xi_t, i, i, t)] \le (-\alpha_1 + \alpha_4 \beta) e^{\beta t} \|\xi(t)\|^2 \\
+ \alpha_3 \mu \lambda_{max}(Z) \beta e^{\beta t} \int_{t-\mu}^{t} \|x(s - \tau_{r_s}(s))\|^2 ds \\
+ \beta e^{\beta t} \big[(\mu \eta + 1) \lambda_{max}(Q) + \alpha_2 \mu \lambda_{max}(Z) \big] \int_{t-\mu}^{t} \|x(s)\|^2 ds,
\end{aligned} \tag{2.25}$$

where $\alpha_4 = \max_{i \in \mathscr{S}} \{\lambda_{max}(P_{ii})\}$.

Using Dynkin's formula [18], for any $T > 0, \beta > 0$, and each $r_t^o = r_t = i,\ i \in \mathscr{S}$, it follows that

$$\begin{aligned}
& E\big\{ e^{\beta T} V(\xi_T, r_t^o, r_t, T) \,\big|\, \xi_0, r_0^o, r_0, 0 \big\} \\
& = V(\xi_0, r_0^o, r_0, 0) + E\Big\{ \int_0^T \mathfrak{L}[e^{\beta s} V(\xi_s, i, i, s)] ds \,\Big|\, \xi_0, r_0^o, r_0, 0 \Big\}.
\end{aligned}$$

Since the initial time values $x(0) = x_0, r_0$ and r_0^o are deterministic, ξ_0 is also deterministic. Substituting (2.25) into above gives

$$\begin{aligned}
& E\big\{ e^{\beta T} V(\xi_T, r_t^o, r_t, T) \big\} \\
& \le V(\xi_0, r_0^o, p_0, 0) + E\bigg\{ (-\alpha_1 + \alpha_4 \beta) \int_0^T e^{\beta t} \|\xi(t)\|^2 dt \\
& + \beta \big[(\mu \eta + 1) \lambda_{max}(Q) + \alpha_2 \mu \lambda_{max}(Z) \big] \int_0^T e^{\beta t} \int_{t-\mu}^{t} \|x(s)\|^2 ds dt \\
& + \alpha_3 \mu \lambda_{max}(Z) \beta \int_0^T e^{\beta t} \int_{t-\mu}^{t} \|x(s - \tau_{r_s}(s))\|^2 ds dt \bigg\}.
\end{aligned} \tag{2.26}$$

Let $\overline{\theta} = t - \tau_i(t)$. The following inequalities

$$\begin{cases} \dot{\tau}_i(t) = \dfrac{d\tau_i(t)}{dt} \leq h_i < 1, \\ dt \leq \dfrac{1}{1-h_i} d\overline{\theta}, \end{cases} \tag{2.27}$$

yields

$$\begin{aligned} &\int_0^T e^{\beta t} \int_{t-\mu}^t \|x(s)\|^2 ds dt \\ &\leq \int_{-\mu}^0 \mu e^{\beta(s+\mu)} \|x(s)\|^2 ds + \int_0^{T-\mu} \mu e^{\beta(s+\mu)} \|x(s)\|^2 ds + \int_{T-\mu}^T \mu e^{\beta(s+\mu)} \|x(s)\|^2 ds \\ &= \mu \int_{-\mu}^T e^{\beta(t+\mu)} \|x(t)\|^2 dt \leq \mu \int_{-\mu}^T e^{\beta(t+\mu)} \|\xi(t)\|^2 dt, \end{aligned} \tag{2.28}$$

$$\begin{aligned} &\int_0^T e^{\beta t} \int_{t-\mu}^t \|x(s - \tau_{r_s}(s))\|^2 ds dt \\ &\leq \int_{-\mu}^0 \mu e^{\beta(s+\mu)} \|x(s - \tau_{r_s}(s))\|^2 ds + \int_0^{T-\mu} \mu e^{\beta(s+\mu)} \|x(s - \tau_{r_s}(s))\|^2 ds \\ &\quad + \int_{T-\mu}^T \mu e^{\beta(s+\mu)} \|x(s - \tau_{r_s}(s))\|^2 ds \\ &= \mu \int_{-\mu}^T e^{\beta(t+\mu)} \|x(t - \tau_{r_t}(t))\|^2 dt \leq \frac{1}{1-h_i} \mu \int_{-2\mu}^T e^{\beta(\overline{\theta}+2\mu)} \|x(\overline{\theta})\|^2 d\overline{\theta} \\ &= \frac{1}{1-h_i} \mu \int_{-2\mu}^T e^{\beta(t+2\mu)} \|x(t)\|^2 dt \leq \frac{1}{1-h_i} \mu \int_{-2\mu}^T e^{\beta(t+2\mu)} \|\xi(t)\|^2 dt. \end{aligned} \tag{2.29}$$

Substituting (2.28) and (2.29) into (2.26) leads to

$$\begin{aligned} &E\{e^{\beta T} V(\xi_T, r_t^o, r_t, T)\} \\ &\leq V(\xi_0, r_0^o, p_0, 0) + E\Big\{(-\alpha_1 + \alpha_4 \beta) \int_0^T e^{\beta t} \|\xi(t)\|^2 dt + \beta\big[(\mu\eta + 1)\lambda_{max}(Q) \\ &\quad + \alpha_2 \mu \lambda_{max}(Z)\big]\mu \int_{-\mu}^T e^{\beta(t+\mu)} \|\xi(t)\|^2 dt + \frac{\alpha_3 \mu^2 \lambda_{max}(Z)\beta}{1-h_i} \int_{-2\mu}^T e^{\beta(t+2\mu)} \|\xi(t)\|^2 dt\Big\} \\ &\leq V(\xi_0, r_0^o, r_0, 0) + E\Big\{\alpha_5 \beta e^{\beta\mu} \int_{-\mu}^0 \|\xi(t)\|^2 dt + \alpha_6 \beta e^{2\beta\mu} \int_{-2\mu}^0 \|\xi(t)\|^2 dt \\ &\quad + [-\alpha_1 + \alpha_4 \beta + \alpha_5 \beta e^{\beta\mu} + \alpha_6 \beta e^{2\beta\mu}] \int_0^T e^{\beta t} \|\xi(t)\|^2 dt\Big\}, \end{aligned}$$

where $\alpha_5 = \big[(\mu\eta + 1)\lambda_{max}(Q) + \alpha_2 \mu \lambda_{max}(Z)\big]\mu$, and $\alpha_6 = \frac{\alpha_3 \mu^2 \lambda_{max}(Z)}{1-h_i}$.

Choose $\beta > 0$ such that

$$-\alpha_1 + \alpha_4\beta + \alpha_5\beta e^{\beta\mu} + \alpha_6\beta e^{2\beta\mu} \leq 0.$$

Then, we have

$$E\left\{e^{\beta T} V(\xi_T, r_t^o, r_t)\right\} \leq c, \tag{2.30}$$

where $c = V(\xi_0, r_0^o, p_0, 0) + E\left\{\alpha_5\beta e^{\beta\mu}\int_{-\mu}^{0}\|\xi(t)\|^2 dt + \alpha_6\beta e^{2\beta\mu}\int_{-2\mu}^{0}\|\xi(t)\|^2 dt\right\}$.

Hence, the LMIs $\Xi_{ii} < 0$, $\Gamma_{ii} \geq 0$ guarantee that the nominal time-delayed jump linear system Σ_0 is exponentially stable in mean square, for $r_t^o = r_t = i, \forall i \in \mathscr{S}$.

Case II. $r_t^o = j, r_t = i$, and $j \neq i$

Following similar lines as in the proof of Case I, we obtain

$$\begin{aligned}
&\mathfrak{L}V(x_t, j, i, t)\\
&\leq \xi^T(t)\left[\widehat{A}_{1ji}^T P_{ji} + P_{ji}\widehat{A}_{1ji} + q_{ji}^0(P_{ii} - P_{ji})\right]\xi(t)\\
&+ \xi^T(t)P_{ji}\widehat{A}_{2i}x(t-\tau_i(t)) + x^T(t-\tau_i(t))\widehat{A}_{2i}^T P_{ji}\xi(t)\\
&+ (1+\eta\mu)\xi^T(t)I_0^T Q I_0\xi(t) - (1-h_i)x^T(t-\tau_i(t))Qx(t-\tau_i(t))\\
&+ \mu\dot{\xi}^T(t)I_0^T Z I_0\dot{\xi}(t) - \int_{t-\tau_i(t)}^{t}\dot{x}^T(s)Z\dot{x}(s)ds\\
&\leq \xi^T(t)\Xi_{ji}\xi(t) - \int_{t-\tau_i(t)}^{t}\chi^T(t,s)\Gamma_{ji}\chi(t,s)ds,
\end{aligned} \tag{2.31}$$

where

$$\Xi_{ji} = \begin{bmatrix} \widehat{\Phi}_{11} + \mu\widehat{A}_{1ji}^T I_0^T Z I_0\widehat{A}_{1ji} & \widehat{\Phi}_{12} + \mu\widehat{A}_{1ji}^T I_0^T Z I_0\widehat{A}_{2i} \\ \widehat{\Phi}_{12}^T + \mu\widehat{A}_{2i}^T I_0^T Z I_0\widehat{A}_{1ji} & \widehat{\Phi}_{22} + \widehat{A}_{2i}^T I_0^T Z I_0\widehat{A}_{2i} \end{bmatrix},$$

and the LMIs $\Xi_{ji} < 0$, $\Gamma_{ji} \geq 0$ guarantee that the nominal time-delayed jump linear system Σ_0 is exponentially stable in mean square, for $r_t^o = j, r_t = i$, and $j \neq i$, $\forall j, i \in S$.

Applying the Schur complement, one sees that for any $i, j \in S$, $\Xi_{ji} < 0$ implies

$$\begin{bmatrix} \widehat{\Phi}_{11} & \widehat{\Phi}_{12} & \mu\widehat{A}_{1ji}^T I_0^T Z \\ \widehat{\Phi}_{12}^T & \widehat{\Phi}_{22} & \mu\widehat{A}_{2i}^T I_0^T Z \\ \mu Z I_0\widehat{A}_{1ji} & \mu Z I_0\widehat{A}_{2i} & -\mu Z \end{bmatrix} < 0, \tag{2.32}$$

which is equivalent to the following condition:

$$\begin{bmatrix} \Phi_{11} & C_i^T B_{3j}^T P_{2ji} + \mu X_{1ii}^2 & \Phi_{13} & \mu A_{1i}^T Z \\ P_{2ji} B_{3j} C_i + \mu X_{1ii}^{2^T} & A_{3j}^T P_{2ji} + P_{2ji} A_{3j} + \Phi_{22} & \mu X_{2ii}^2 & 0 \\ \Phi_{13}^T & \mu X_{2ii}^{2^T} & \Phi_{33} & \mu A_{2i}^T Z \\ \mu Z A_{1i} & 0 & \mu Z A_{2i} & -\mu Z \end{bmatrix}$$
$$+ \begin{bmatrix} P_{1ji} B_{1i} \\ 0 \\ 0 \\ \mu Z B_{1i} \end{bmatrix} I \begin{bmatrix} 0 & K_j & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ K_j^T \\ 0 \\ 0 \end{bmatrix} I \begin{bmatrix} B_{1i}^T P_{1ji} & 0 & 0 & \mu B_{1i}^T Z \end{bmatrix} < 0. \quad (2.33)$$

By Lemma 2.2, a sufficient condition guaranteeing (2.33) is that there exists a positive number $\rho_{ji} > 0$ such that

$$\rho_{ji} \begin{bmatrix} \Phi_{11} & C_i^T B_{3j}^T P_{2ji} + \mu X_{1ii}^2 & \Phi_{13} & \mu A_{1i}^T Z \\ P_{2ji} B_{3j} C_i + \mu X_{1ii}^{2^T} & A_{3j}^T P_{2ji} + P_{2ji} A_{3j} + \Phi_{22} & \mu X_{2ii}^2 & 0 \\ \Phi_{13}^T & \mu X_{2ii}^{2^T} & \Phi_{33} & \mu A_{2i}^T Z \\ \mu Z A_{1i} & 0 & \mu Z A_{2i} & -\mu Z \end{bmatrix}$$
$$+\rho_{ji}^2 \begin{bmatrix} P_{1ji} B_{1i} \\ 0 \\ 0 \\ \mu Z B_{1i} \end{bmatrix} I \begin{bmatrix} B_{1i}^T P_{1ji} & 0 & 0 & \mu B_{1i}^T Z \end{bmatrix} + \begin{bmatrix} 0 \\ K_i^T \\ 0 \\ 0 \end{bmatrix} I \begin{bmatrix} 0 & K_i & 0 & 0 \end{bmatrix} < 0. \quad (2.34)$$

Replacing $\rho_{ji} P_{1ji}$, $\rho_{ji} P_{2ji}$, $\rho_{ji} Q$, $\rho_{ji} Z$, $\rho_{ji} X_{ji}$, $\rho_{ji} Y_{ji}$ and $\rho_{ji} T_{ji}$ with P_{1ji}, P_{2ji}, Q, Z, X_{ji}, Y_{ji} and T_{ji}, respectively, and applying the Schur complement shows that (2.34) is equivalent to

$$\mathbf{W}_{ji} = \begin{bmatrix} \Phi_{11} & C_i^T V_{ji} + \mu X_{1ii}^2 & \Phi_{13} & \mu A_{1i}^T Z & P_{1ji} B_{1i} & 0 \\ V_{ji}^T C_i + \mu X_{1ii}^{2^T} & U_{ji} + U_{ji}^T + \Phi_{22} & \mu X_{2ii}^2 & 0 & 0 & K_j^T \\ \Phi_{13}^T & \mu X_{2ii}^{2^T} & \Phi_{33} & \mu A_{2i}^T Z & 0 & 0 \\ \mu Z A_{1i} & 0 & \mu Z A_{2i} & -\mu Z & \mu Z B_{1i} & 0 \\ B_{1i}^T P_{1ji} & 0 & 0 & \mu B_{1i}^T Z & -I & 0 \\ 0 & K_j & 0 & 0 & 0 & -I \end{bmatrix} < 0 \quad (2.35)$$

with $j, i \in \mathscr{S}$. Hence, the LMIs (2.16) (2.35) guarantee that the nominal time-delayed jump linear system Σ_0 is exponentially stable in mean square for $r_t^o = j, r_t = i, \ \forall j, i \in S$.

Then, for the uncertain time-delayed jump linear system (2.10) without disturbance, replacing A_{1i}, A_{2i}, B_{1i} and K_j in (2.35) with $A_{1i} + H_{1i} F_i(t) E_{1i}$, $A_{2i} + H_{1i} F_i(t) E_{2i}$, $B_{1i} + H_{1i} F_i(t) E_{3i}$ and $K_j + \alpha_i \phi_i(t) K_j$, we can obtain that (2.35) for system (2.10) is equivalent to the following condition:

$$
\begin{aligned}
&\mathbf{W}_{ji} + \begin{bmatrix} P_{1ji}H_{1i} \\ 0 \\ 0 \\ \mu Z H_{1i} \\ 0 \\ 0 \end{bmatrix} F_i(t) \begin{bmatrix} E_{1i}^T \\ 0 \\ E_{2i}^T \\ 0 \\ E_{3i}^T \\ 0 \end{bmatrix}^T + \begin{bmatrix} E_{1i}^T \\ 0 \\ E_{2i}^T \\ 0 \\ E_{3i}^T \\ 0 \end{bmatrix} F_i(t) \begin{bmatrix} P_{ji}H_{1i} \\ 0 \\ 0 \\ \mu Z H_{1i} \\ 0 \\ 0 \end{bmatrix}^T \\
&+ \begin{bmatrix} 0 \\ V_{ji}^T H_{2i} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} F_i(t) \begin{bmatrix} E_{4i}^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} E_{4i}^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} F_i(t) \begin{bmatrix} 0 \\ V_{ji}^T H_{2i} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \\
&+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha_i \end{bmatrix} \phi_i(t) \begin{bmatrix} 0 \\ K_j^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} 0 \\ K_j^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \phi_i^T(t) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha_i \end{bmatrix}^T < 0. \qquad (2.36)
\end{aligned}
$$

By Lemma 2.2, a sufficient condition guaranteeing (2.36) is that there exist positive numbers $\rho_{1ji} > 0$, $\rho_{2ji} > 0$, $\rho_{3ji} > 0$ such that

$$
\begin{aligned}
&\mathbf{W}_{ji} + \rho_{3ji}^{-1} \begin{bmatrix} P_{1ji}H_{1i} \\ 0 \\ 0 \\ \mu Z H_{1i} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} P_{1ji}H_{1i} \\ 0 \\ 0 \\ \mu Z H_{1i} \\ 0 \\ 0 \end{bmatrix}^T + \rho_{3ji} \begin{bmatrix} E_{1i}^T \\ 0 \\ E_{2i}^T \\ 0 \\ E_{3i}^T \\ 0 \end{bmatrix} \begin{bmatrix} E_{1i}^T \\ 0 \\ E_{2i}^T \\ 0 \\ E_{3i}^T \\ 0 \end{bmatrix}^T \\
&+ \rho_{2ji}^{-1} \begin{bmatrix} 0 \\ V_{ji}^T H_{2i} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ V_{ji}^T H_{2i} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \rho_{2ji} \begin{bmatrix} E_{4i}^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} E_{4i}^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \\
&+ \rho_{1ji} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha_i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha_i \end{bmatrix}^T + \rho_{1ji}^{-1} \begin{bmatrix} 0 \\ K_j^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ K_j^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T < 0. \qquad (2.37)
\end{aligned}
$$

With the Schur complement one can show that (2.15) is equivalent to (2.37) for all $r_t^o = j, r_t = i, \ \forall j, i \in \mathscr{S}$. This completes the proof.

Remark 2.3 It can be seen that the condition in (2.32) is nonlinear in the design parameters A_{3j}, B_{3j}, K_j and P_{ji}. In non-delayed systems, these types of nonlinearities have been eliminated by some appropriate change of control variables with the general form of P_{ji} as follows [13, 21]:

$$P_{ji} = \begin{bmatrix} P_{1ji} & P_{2ji} \\ P_{2ji}^T & P_{3ji} \end{bmatrix}, \quad \forall j, i \in \mathscr{S}. \tag{2.38}$$

To deal with the output feedback control problem for time-delay systems, there are always some parameters coupled with their inverse which is required to be fixed a priori, see, e.g., [10, 26]. In this chapter, if we partition P_{ji} as (2.38) and use the linearizing change of variable approach as in [26] for condition (2.32), the design parameters Y_{ji}, X_{1ji}^1, Y_{ji}^{-1}, $X_{1ji}^{1^{-1}}$ will occur in the same inequality.

Then, if we were to transfer the control design problem into the framework of LMI, we have to fix these parameters a priori, which makes the obtaining of the optimal relationships between the terms in the Newton-Leibniz formula (2.20) and (2.21) almost impossible.

To obtain an easier design technique, we choose P_{ji} to be diagonal block matrices

$$P_{ji} = \begin{bmatrix} P_{1ji} & 0 \\ 0 & P_{2ji} \end{bmatrix}, \quad \forall j, i \in \mathscr{S}.$$

It is reasonable to choose Lyapunov parameters P_{1ji} for plant states $x(t)$ and P_{2ji} for control systems states $\widehat{x}(t)$, respectively. We can obtain the optimal free weighting matrices by solving the corresponding linear matrix inequalities without the need to fix any design parameters, leading to less conservative results.

2.4 Robust H_∞ Disturbance Attenuation

In this section, we consider robust H_∞ disturbance attenuation for the time-delayed uncertain jump linear systems (2.10).

Theorem 2.2 *The time-delayed uncertain jump linear systems (2.10) is stochastically stable with γ-disturbance H_∞ attenuation (2.11), and the output feedback control law (2.8) is robust if there exist symmetric positive-definite matrices P_{1ji}, P_{2ji}, Q, Z, symmetric positive semi-definite matrices $\overline{X}_{ji} \geq 0$, constants $\rho_{1ji} > 0$, $\rho_{2ji} > 0$, $\rho_{3ji} > 0$ and appropriately dimensioned matrices K_j, Y_{ji}, T_{ji}, N_{ji} such that*

$$\begin{bmatrix} \overline{L}_1 & \mu A_{1i}^T\widehat{Z} & \widehat{P}_{1ji}B_{1i}+\rho_{3ji}E_{1i}^TE_{3i} & 0 & C_i^T & \widehat{P}_{1ji}H_{1i} & 0 & 0 \\ \overline{L}_2 & 0 & 0 & K_j^T & 0 & 0 & V_{ji}^TH_{2i} & K_j^T \\ \overline{L}_3 & \mu A_{2i}^T\widehat{Z} & \rho_{3ji}E_{2i}^TE_{3i} & 0 & 0 & 0 & 0 & 0 \\ \overline{L}_4 & \mu B_{2i}^T\widehat{Z} & 0 & 0 & 0 & 0 & 0 & 0 \\ \overline{L}_5 & -\mu\widehat{Z} & \mu\widehat{Z}B_{1i} & 0 & 0 & \mu\widehat{Z}H_{1i} & 0 & 0 \\ \overline{L}_6 & \mu B_{1i}^T\widehat{Z} & -I+\rho_{3ji}E_{3i}^TE_{3i} & 0 & 0 & 0 & 0 & 0 \\ \overline{L}_7 & 0 & 0 & -I+\rho_{1ji}\alpha_i^2 I & 0 & 0 & 0 & 0 \\ \overline{L}_8 & 0 & 0 & 0 & -\rho_{4ji}I & 0 & H_{2i} & 0 \\ \overline{L}_9 & \mu H_{1i}^T\widehat{Z} & 0 & 0 & 0 & -\rho_{3ji}I & 0 & 0 \\ \overline{L}_{10} & 0 & 0 & 0 & H_{2i}^T & 0 & -\rho_{2ji}I & 0 \\ \overline{L}_{11} & 0 & 0 & 0 & 0 & 0 & 0 & -\rho_{1ji}I \end{bmatrix} < 0, \tag{2.39}$$

$$\overline{\Gamma}_{ji} = \begin{bmatrix} \widehat{X}_{11ji} & \widehat{X}_{12ji} & \widehat{X}_{13ji} & I_0^T\widehat{Y}_{ji} \\ \widehat{X}_{12ji}^T & \widehat{X}_{22ji} & \widehat{X}_{23ji} & \widehat{T}_{ji} \\ \widehat{X}_{13ji}^T & \widehat{X}_{23ji}^T & \widehat{X}_{33ji} & \widehat{N}_{ji} \\ \widehat{Y}_{ji}^TI_0 & \widehat{T}_{ji}^T & \widehat{N}_{ji}^T & \widehat{Z} \end{bmatrix} \geq 0, \quad \forall i, j \in \mathscr{S}, \tag{2.40}$$

where

$$\begin{bmatrix} \overline{L}_1 \\ \overline{L}_2 \\ \overline{L}_3 \\ \overline{L}_4 \\ \overline{L}_5 \\ \overline{L}_6 \\ \overline{L}_7 \\ \overline{L}_8 \\ \overline{L}_9 \\ \overline{L}_{10} \\ \overline{L}_{11} \end{bmatrix} = \begin{bmatrix} \Psi_{11}+\rho_{3ji}E_{1i}^TE_{1i}+\rho_{2ji}E_{4i}^TE_{4i} & C_i^TV_{ji}+\mu\widehat{X}_{11ji}^2 & \Psi_{13}+\rho_{3ji}E_{1i}^TE_{2i} & \Psi_{14} \\ \mu\widehat{X}_{11ji}^{2^T}+V_{ji}^TC_i & U_i+U_i^T+\Psi_{22} & \mu\widehat{X}_{12ji}^2 & \mu\widehat{X}_{13ji}^2 \\ \Psi_{13}^T+\rho_{3ji}E_{2i}^TE_{1i} & \mu\widehat{X}_{12ji}^{2^T} & \Psi_{33}+\rho_{3ji}E_{2i}^TE_{2i} & \Psi_{34} \\ \Psi_{14}^T & \mu\widehat{X}_{13ji}^{2^T} & \Psi_{34}^T & \Psi_{44}-\gamma^2 I \\ \mu\widehat{Z}A_{1i} & 0 & \mu\widehat{Z}A_{2i} & \mu\widehat{Z}B_{2i} \\ B_{1i}^T\widehat{P}_{1ji}+\rho_{3ji}E_{3i}^TE_{1i} & 0 & \rho_{3ji}E_{3i}^TE_{2i} & 0 \\ 0 & K_j & 0 & 0 \\ C_i & 0 & 0 & 0 \\ H_{1i}^T\widehat{P}_{1ji} & 0 & 0 & 0 \\ 0 & H_{2i}^TV_{ji} & 0 & 0 \\ 0 & K_j & 0 & 0 \end{bmatrix},$$

$$\overline{X}_{ji} = \begin{bmatrix} \overline{X}_{11ji} & \overline{X}_{12ji} & \overline{X}_{13ji} \\ \overline{X}_{12ji}^T & \overline{X}_{22ji} & \overline{X}_{23ji} \\ \overline{X}_{13ji}^T & \overline{X}_{23ji}^T & \overline{X}_{33ji} \end{bmatrix} = \begin{bmatrix} \overline{X}_{11ji}^1 & \overline{X}_{11ji}^2 & \overline{X}_{12ji}^1 & \overline{X}_{13ji}^1 \\ \overline{X}_{11ji}^{2^T} & \overline{X}_{11ji}^3 & \overline{X}_{12ji}^2 & \overline{X}_{13ji}^2 \\ \overline{X}_{12ji}^{1^T} & \overline{X}_{12ji}^{2^T} & \overline{X}_{22ji} & \overline{X}_{23ji} \\ \overline{X}_{13ji}^{1^T} & \overline{X}_{13ji}^{2^T} & \overline{X}_{23ji}^T & \overline{X}_{33ji} \end{bmatrix},$$

with

$$\Psi_{11} = \begin{cases} (\text{if} \quad j = i) \\ A_{1i}^T \widehat{P}_{1ii} + \widehat{P}_{1ii} A_{1i} + \widehat{Y}_{ii} + \widehat{Y}_{ii}^T + (1+\eta\mu)\widehat{Q} + \mu \widehat{X}_{11ii}^1 + \sum_{j=1}^N \pi_{ij} \widehat{P}_{1ij} + \sum_{j=1}^N \pi_{ij}^1 \widehat{P}_{1ji}, \\ (\text{if} \quad j \neq i) \\ A_{1i}^T \widehat{P}_{1ji} + \widehat{P}_{1ji} A_{1i} + \widehat{Y}_{ji} + \widehat{Y}_{ji}^T + (1+\eta\mu)\widehat{Q} + \mu \widehat{X}_{11ji}^1 + \pi_{ji}^0 (\widehat{P}_{1ii} - \widehat{P}_{1ji}), \end{cases}$$

$$\Psi_{22} = \begin{cases} \sum_{j=1}^N \pi_{ij} \widehat{P}_{2ij} + \sum_{j=1}^N \pi_{ij}^1 \widehat{P}_{2ji} + \mu \widehat{X}_{11ii}^3, & \text{if} \;\; j = i \\ \pi_{ji}^0 (\widehat{P}_{2ii} - \widehat{P}_{2ji}) + \mu \widehat{X}_{11ji}^3, & \text{if} \;\; j \neq i \end{cases}$$

$$\Psi_{13} = \widehat{P}_{1ji} A_{2i} - \widehat{Y}_{ji} + \widehat{T}_{ji}^T + \mu \widehat{X}_{12ji}^1, \qquad \Psi_{14} = \widehat{P}_{1ji} B_{2i} + \widehat{N}_{ji}^T + \mu \widehat{X}_{13ji}^1,$$

$$\Psi_{33} = -\widehat{T}_{ji} - \widehat{T}_{ji}^T - (1-h_i)\widehat{Q} + \mu \widehat{X}_{22ji}, \quad \Psi_{34} = -\widehat{N}_{ji}^T + \mu \widehat{X}_{23ji},$$

$$\Psi_{44} = \mu \widehat{X}_{33ji}, \quad \eta = \max_{i \in \mathscr{S}} \{|\pi_{ii}|\}, \quad V_{ji} = B_{3j}^T \widehat{P}_{2ji}, \quad U_{ji} = A_{3j}^T \widehat{P}_{2ji},$$

$$\begin{bmatrix} \widehat{P}_{1ji} & \widehat{P}_{2ji} & \widehat{Q} & \widehat{Z} & \widehat{Y}_{ji} & \widehat{T}_{ji} & \widehat{N}_{ji} \end{bmatrix} = \rho_{4ji}^{-1} \begin{bmatrix} P_{1ji} & P_{2ji} & Q & Z & Y_{ji} & T_{ji} & N_{ji} \end{bmatrix},$$

$$\begin{bmatrix} \widehat{X}_{11ji}^1 & \widehat{X}_{11ji}^2 & \widehat{X}_{12ji}^1 & \widehat{X}_{13ji}^1 \\ \widehat{X}_{11ji}^{2^T} & \widehat{X}_{11ji}^3 & \widehat{X}_{12ji}^2 & \widehat{X}_{13ji}^2 \\ \widehat{X}_{12ji}^{1^T} & \widehat{X}_{12ji}^{2^T} & \widehat{X}_{22ji} & \widehat{X}_{23ji} \\ \widehat{X}_{13ji}^{1^T} & \widehat{X}_{13ji}^{2^T} & \widehat{X}_{23ji}^T & \widehat{X}_{33ji} \end{bmatrix} = \rho_{4ji}^{-1} \begin{bmatrix} \overline{X}_{11ji}^1 & \overline{X}_{11ji}^2 & \overline{X}_{12ji}^1 & \overline{X}_{13ji}^1 \\ \overline{X}^{2^T}_{11ji} & \overline{X}_{11ji}^3 & \overline{X}_{12ji}^2 & \overline{X}_{13ji}^2 \\ \overline{X}^{1^T}_{12ji} & \overline{X}^{2^T}_{12ji} & \overline{X}_{22ji} & \overline{X}_{23ji} \\ \overline{X}^{1^T}_{13ji} & \overline{X}^{2^T}_{13ji} & \overline{X}_{23ji}^T & \overline{X}_{33ji} \end{bmatrix}.$$

Proof For the nominal time-delayed jump linear system Σ_1 with disturbance:

$$\Sigma_1 : \quad \begin{cases} \dot{\xi}(t) = \widehat{A}_1(r_t^o, r_t)\xi(t) + \widehat{A}_2(r_t) I_0 \xi(t - \tau_{r_t}(t)) + \widehat{B}_2(r_t) w(t), \\ z(t) = [C(r_t) + \Delta_C(r_t, t)] I_0 \xi(t), \\ I_0 \xi(s) = f(s), \quad r_s = r_0, \quad s \in [-\mu, \;\; 0], \end{cases} \tag{2.41}$$

where

$$\widehat{B}_2(r_t) = \begin{bmatrix} B_2(r_t) \\ 0 \end{bmatrix}.$$

Let $\overline{\zeta}^T(t) = [\xi^T(t) \;\; x^T(t - \tau_i(t)) \;\; w^T(t)]$. Take the Lyapunov function candidate as (2.18), and employ the following conditions

$$\mu \overline{\zeta}^T(t) \overline{X}(r_t^o, r_t) \overline{\zeta}(t) - \int_{t-\tau_{r_t}(t)}^t \overline{\zeta}^T(t) \overline{X}(r_t^o, r_t) \overline{\zeta}(t) ds \geq 0,$$

$$2\left[\xi^T(t) I_0^T Y(r_t^o, r_t) + x^T(t - \tau_{r_t}(t)) T(r_t^o, r_t) + w^T(t) N(r_t^o, r_t)\right] \times \left[I_0 \xi(t) - \int_{t-\tau_{r_t}(t)}^t \dot{x}(s) ds - x(t - \tau_{r_t}(t))\right] = 0, \tag{2.42}$$

we can then obtain

$$\mathfrak{L}V(x_t, j, i, t) \leq \overline{\zeta}^T(t)\overline{\Xi}_{ji}\overline{\zeta}(t) - \int_{t-\tau_i(t)}^{t} \overline{\chi}^T(t,s)\overline{\Gamma}_{ji}\overline{\chi}(t,s)ds, \tag{2.43}$$

where

$$\overline{\Xi}_{ji}=\begin{bmatrix} \overline{\Psi}_{11} + \mu\widehat{A}_{1ji}^T I_0^T Z I_0 \widehat{A}_{1ji} & \overline{\Psi}_{12} + \mu\widehat{A}_{1ji}^T I_0^T Z I_0 \widehat{A}_{2i} & \overline{\Psi}_{13} + \widehat{A}_{1ji}^T I_0^T Z I_0 \widehat{B}_{2i} \\ \overline{\Psi}_{12}^T + \mu\widehat{A}_{2i}^T I_0^T Z I_0 \widehat{A}_{1ji} & \overline{\Psi}_{22} + \mu\widehat{A}_{2i}^T I_0^T Z I_0 \widehat{A}_{2i} & \overline{\Psi}_{23} + \mu\widehat{A}_{2i}^T I_0^T Z I_0 \widehat{B}_{2i} \\ \overline{\Psi}_{13}^T + \mu\widehat{B}_{2i}^T I_0^T Z I_0 \widehat{A}_{1ji} & \overline{\Psi}_{23}^T + \mu\widehat{B}_{2i}^T I_0^T Z I_0 \widehat{A}_{2i} & \overline{\Psi}_{33} + \mu\widehat{B}_{2i}^T I_0^T Z I_0 \widehat{B}_{2i} \end{bmatrix}$$

$$\overline{\Psi}_{11} = \begin{cases} \widehat{A}_{1ii}^T P_{ii} + P_{ii}\widehat{A}_{1ii} + I_0^T Y_{ii} I_0 + I_0 Y_{ii}^T I_0^T + (1+\eta\mu) I_0^T Q I_0 & \\ \quad +\mu\overline{X}_{11ii} + \sum_{j=1}^{N} \pi_{ij} P_{ij} + \sum_{j=1}^{N} \pi_{ij}^1 P_{ji}, & \text{if } j = i \\ \widehat{A}_{1ji}^T P_{ji} + P_{ji}\widehat{A}_{1ji} + I_0^T Y_{ji} I_0 + I_0 Y_{ji}^T I_0^T + (1+\eta\mu) I_0^T Q I_0 & \\ \quad +\mu\overline{X}_{11ji} + q_{ji}^0 (P_{ii} - P_{ji}), & \text{if } j \neq i \end{cases}$$

$$\overline{\Psi}_{12} = P_{ji}\widehat{A}_{2i} - I_0^T Y_{ji} + I_0^T T_{ji}^T + \mu\overline{X}_{12ji},$$
$$\overline{\Psi}_{22} = -T_{ji} - T_{ji}^T - (1-h_i)Q + \mu\overline{X}_{22ji},$$
$$\overline{\Psi}_{13} = P_{ji}\widehat{B}_{2i} + I_0^T N_{ji}^T + \mu\overline{X}_{13ji}, \ \overline{\Psi}_{23} = -N_{ji}^T + \mu\overline{X}_{23ji}, \ \overline{\Psi}_{33} = \overline{X}_{33ji},$$
$$\eta = \max_{i\in\mathscr{S}}\{|\pi_{ii}|\}, \overline{\chi}^T(t,s) = [\xi^T(t) \ \ x^T(t-\tau_i(t)) \ \ w^T(t) \ \ \dot{x}^T(s)].$$

Using Dynkin's formula again [18], we obtain

$$E\left\{\int_0^T \mathfrak{L}V(x_s, r_s^o, r_s, s)ds\right\} = E\{V(x_T, r_t^o, r_t, T)\} - E\{V(x_0, r_0^o, r_0, 0)\}.$$

Under the zero initial condition ($x(0) = 0$), we have

$$E\{V(x_0, r_0^o, r_0, 0)\} = 0.$$

Thus, for any $w(t) \in L_2[0 \ \ \infty)$, one sees that

$$\begin{aligned} J &= E\left\{\int_0^T \left[z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \mathfrak{L}V(x_t, r_t^o, r_t, t)\right]dt\right\} - E\{V(x_T, r_t^o, r_t, T)\} \\ &\leq E\left\{\int_0^T \left[z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \mathfrak{L}V(x_t, r_t^o, r_t, t)\right]dt\right\}. \end{aligned} \tag{2.44}$$

Substituting (2.43) into the above inequality gives

$$
\begin{aligned}
J \le E\Big\{ \int_0^T \Big[\overline{\zeta}^T(t)\Big(\overline{\Xi}_{ji} + \begin{bmatrix} I_0^T C_i^T C_i I_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\gamma^2 I \end{bmatrix}\Big)\overline{\zeta}(t) \\
- \int_{t-\tau_i(t)}^{t} \overline{\chi}^T(t,s)\overline{\Gamma}_{ji}\overline{\chi}(t,s)ds\Big]dt\Big\}.
\end{aligned}
$$

By Lemma 2.2 and the Schur complement we obtain (2.40), and

$$
\begin{bmatrix}
\Psi_{11} + \rho_{4ji}^{-1} C_i^T C_i & C_i^T B_{3j}^T \widehat{P}_{2ji} + \mu \widehat{X}^2_{11ji} & \Psi_{13} & \Psi_{14} & \mu A_{1i}^T \widehat{Z} & \widehat{P}_{1ji} B_{1i} & 0 \\
\widehat{P}_{2ji} B_{3j} C_i + \mu \widehat{X}^{2^T}_{11ji} & A_{3j}^T \widehat{P}_{2ji} + \widehat{P}_{2ji} A_{3j} + \Psi_{22} & \mu \widehat{X}^2_{12ji} & \mu \widehat{X}^2_{13ji} & 0 & 0 & K_j^T \\
\Psi_{13}^T & \mu \widehat{X}^{2^T}_{12ji} & \Psi_{33} & \Psi_{34} & \mu A_{2i}^T \widehat{Z} & 0 & 0 \\
\Psi_{14}^T & \mu \widehat{X}^{2^T}_{13ji} & \Psi_{34}^T & \Psi_{44} - \gamma^2 I & \mu B_{2i}^T \widehat{Z} & 0 & 0 \\
\mu \widehat{Z} A_{1i} & 0 & \mu \widehat{Z} A_{2i} & \mu \widehat{Z} B_{2i} & -\mu \widehat{Z} & \widehat{Z} B_{1i} & 0 \\
B_{1i}^T \widehat{P}_{1ji} & 0 & 0 & 0 & \mu B_{1i}^T \widehat{Z} & -I & 0 \\
0 & K_j & 0 & 0 & 0 & 0 & 0
\end{bmatrix} < 0 \quad (2.45)
$$

guarantee $J < 0$ for any $w(t) \neq 0$ (and $w(t) \in L_2[0, \infty)$), which also guarantee γ-disturbance H_∞ attenuation (2.11) of the closed-loop system Σ_1 from $w(t)$ to $z(t)$.

Then, replacing A_{1i}, A_{2i}, B_{1i}, C_i and K_j in (2.45) with $A_{1i} + H_{1i}F_i(t)E_{1i}$, $A_{2i} + H_{1i}F_i(t)E_{2i}$, $B_{1i} + H_{1i}F_i(t)E_{3i}$, $C_i + H_{2i}F_i(t)E_{4i}$ and $K_j + \alpha_i\phi_i(t)K_j$ and using the similar proof of Theorem 2.1, we can easily verify that the control $u(t) = K(r_t^o)x(t)$ guarantees γ-disturbance H_∞ attenuation (2.11) of the closed-loop system (2.10) from $w(t)$ to $z(t)$, if the coupled linear matrix inequalities (2.39) and (2.40) are satisfied. This completes the proof.

In the case that the jumping parameter process can be directly and precisely measured; that is, $r_t = r_t^o$, $\forall t \in [0, \infty)$, the closed-loop system (2.10) is specialized as

$$
\begin{cases}
\dot{\xi}(t) = \widetilde{A}_1(r_t, t)\xi(t) + \widetilde{A}_2(r_t)I_0\xi(t - \tau_{r_t}(t)) + \widetilde{B}_2(r_t)w(t), \\
z(t) = [C(r_t) + \Delta_C(r_t, t)]I_0\xi(t), \\
I_0\xi(s) = f(s), \quad r_s = r_0, \quad s \in [-2\mu, 0],
\end{cases} \quad (2.46)
$$

where

$$\widetilde{A}_{1i} = \begin{bmatrix} A_{1i} + \varDelta_{A_{1i}}(t) & \left(B_{1i} + \varDelta_{B_{1i}}(t)\right)(I + \alpha_i \phi_i(t))\, K_i \\ B_{3i}\left(C_i + \varDelta_{C_i}(t)\right) & A_{3i} \end{bmatrix} \in \mathbb{R}^{2n\times 2n},$$

$$\widetilde{A}_{2i} = \begin{bmatrix} A_{2i} + \varDelta_{A_{2i}}(t) \\ 0 \end{bmatrix} \in \mathbb{R}^{2n\times n},\ \ \widetilde{B}_{2i} = \begin{bmatrix} B_{2i} \\ 0 \end{bmatrix} \in \mathbb{R}^{2n\times m_2},\ \ I_0 = \begin{bmatrix} I & 0 \end{bmatrix} \in \mathbb{R}^{n\times 2n}$$

for each $p_t = i,\ \ \forall i \in \mathscr{S}$.

Then by Theorem 2.2, we have the following corollary.

Corollary 2.1 *The time-delayed uncertain jump linear systems (2.10) is stochastically stable with γ-disturbance H_∞ attenuation (2.11), and the output feedback control law (2.8) is robust if the jumping parameter process can be directly and precisely measured, and there exist symmetric positive-definite matrices P_{1i}, P_{2i}, Q, Z, symmetric positive semi-definite matrices $\widetilde{X}_i \geq 0$, constants $\rho_{1i} > 0$, $\rho_{2i} > 0$, $\rho_{3i} > 0$ and appropriately dimensioned matrices K_i, Y_i, T_i, N_i such that*

$$\begin{bmatrix}
\widetilde{L}_1 & \mu A_{1i}^T \widehat{Z} & \widehat{P}_{1i} B_{1i} + \rho_{3i} E_{1i}^T E_{3i} & 0 & C_i^T & \widehat{P}_{1i} H_{1i} & 0 & 0 \\
\widetilde{L}_2 & 0 & 0 & K_i^T & 0 & 0 & V_i^T H_{2i} & K_i^T \\
\widetilde{L}_3 & \mu A_{2i}^T \widehat{Z} & \rho_{3i} E_{2i}^T E_{3i} & 0 & 0 & 0 & 0 & 0 \\
\widetilde{L}_4 & \mu B_{2i}^T \widehat{Z} & 0 & 0 & 0 & 0 & 0 & 0 \\
\widetilde{L}_5 & -\mu \widehat{Z} & \mu \widehat{Z} B_{1i} & 0 & 0 & \mu \widehat{Z} H_{1i} & 0 & 0 \\
\widetilde{L}_6 & \mu B_{1i}^T \widehat{Z} & -I + \rho_{3i} E_{3i}^T E_{3i} & 0 & 0 & 0 & 0 & 0 \\
\widetilde{L}_7 & 0 & 0 & -I + \rho_{1i} \alpha_i^2 I & 0 & 0 & 0 & 0 \\
\widetilde{L}_8 & 0 & 0 & 0 & -\rho_{4i} I & 0 & H_{2i} & 0 \\
\widetilde{L}_9 & \mu H_{1i}^T \widehat{Z} & 0 & 0 & 0 & -\rho_{3i} I & 0 & 0 \\
\widetilde{L}_{10} & 0 & 0 & 0 & H_{2i}^T & 0 & -\rho_{2i} I & 0 \\
\widetilde{L}_{11} & 0 & 0 & 0 & 0 & 0 & 0 & -\rho_{1i} I
\end{bmatrix} < 0 \qquad (2.47)$$

$$\widetilde{\varGamma}_i = \begin{bmatrix}
\widehat{X}_{11i} & \widehat{X}_{12i} & \widehat{X}_{13i} & I_0^T \widehat{Y}_i \\
\widehat{X}_{12i}^T & \widehat{X}_{22i} & \widehat{X}_{23i} & \widehat{T}_i \\
\widehat{X}_{13i}^T & \widehat{X}_{23i}^T & \widehat{X}_{33i} & \widehat{N}_i \\
\widehat{Y}_i^T I_0 & \widehat{T}_i^T & \widehat{N}_i^T & \widehat{Z}
\end{bmatrix} \geq 0,\quad \forall i \in \mathscr{S}, \qquad (2.48)$$

where

$$\begin{bmatrix} \widetilde{L}_1 \\ \widetilde{L}_2 \\ \widetilde{L}_3 \\ \widetilde{L}_4 \\ \widetilde{L}_5 \\ \widetilde{L}_6 \\ \widetilde{L}_7 \\ \widetilde{L}_8 \\ \widetilde{L}_9 \\ \widetilde{L}_{10} \\ \widetilde{L}_{11} \end{bmatrix} = \begin{bmatrix}
\widetilde{\varPsi}_{11} + \rho_{3i} E_{1i}^T E_{1i} + \rho_{2i} E_{4i}^T E_{4i} & C_i^T V_i + \mu \widehat{X}_{11i}^2 & \widetilde{\varPsi}_{13} + \rho_{3i} E_{1i}^T E_{2i} & \widetilde{\varPsi}_{14} \\
\mu \widehat{X}_{11i}^{2^T} + V_i^T C_i & U_i + U_i^T + \widetilde{\varPsi}_{22} & \mu \widehat{X}_{12i}^2 & \mu \widehat{X}_{13i}^2 \\
\widetilde{\varPsi}_{13}^T + \rho_{3i} E_{2i}^T E_{1i} & \mu \widehat{X}_{12i}^{2^T} & \widetilde{\varPsi}_{33} + \rho_{3i} E_{2i}^T E_{2i} & \widetilde{\varPsi}_{34} \\
\widetilde{\varPsi}_{14}^T & \mu \widehat{X}_{13i}^{2^T} & \widetilde{\varPsi}_{34}^T & \widetilde{\varPsi}_{44} - \gamma^2 I \\
\mu \widehat{Z} A_{1i} & 0 & \mu \widehat{Z} A_{2i} & \mu \widehat{Z} B_{2i} \\
B_{1i}^T \widehat{P}_{1i} + \rho_{3i} E_{3i}^T E_{1i} & 0 & \rho_{3i} E_{3i}^T E_{2i} & 0 \\
0 & K_i & 0 & 0 \\
C_i & 0 & 0 & 0 \\
H_{1i}^T \widehat{P}_{1i} & 0 & 0 & 0 \\
0 & H_{2i}^T V_i & 0 & 0 \\
0 & K_i & 0 & 0
\end{bmatrix},$$

$$\widetilde{X}_i = \begin{bmatrix} \widetilde{X}_{11i} & \widetilde{X}_{12i} & \widetilde{X}_{13i} \\ \widetilde{X}_{12i}^T & \widetilde{X}_{22i} & \widetilde{X}_{23i} \\ \widetilde{X}_{13i}^T & \widetilde{X}_{23i}^T & \widetilde{X}_{33i} \end{bmatrix} = \begin{bmatrix} \widetilde{X}_{11i}^1 & \widetilde{X}_{11i}^2 & \widetilde{X}_{12i}^1 & \widetilde{X}_{13i}^1 \\ \widetilde{X}_{11i}^{2^T} & \widetilde{X}_{11i}^3 & \widetilde{X}_{12i}^2 & \widetilde{X}_{13i}^2 \\ \widetilde{X}_{12i}^{1^T} & \widetilde{X}_{12i}^{2^T} & \widetilde{X}_{22i} & \widetilde{X}_{23i} \\ \widetilde{X}_{13i}^{1^T} & \widetilde{X}_{13i}^{2^T} & \widetilde{X}_{23i}^T & \widetilde{X}_{33i} \end{bmatrix},$$

$$\widetilde{\Psi}_{11} = A_{1i}^T \widehat{P}_{1i} + \widehat{P}_{1i} A_{1i} + \widehat{Y}_i + \widehat{Y}_i^T + (1+\eta\mu)\widehat{Q} + \mu \widehat{X}_{11i}^1 + \sum_{j=1}^{N} \pi_{ij} \widehat{P}_{1j},$$

$$\widetilde{\Psi}_{22} = \sum_{j=1}^{N} \pi_{ij} \widehat{P}_{2j} + \mu \widehat{X}_{11i}^3, \quad \widetilde{\Psi}_{13} = \widehat{P}_{1i} A_{2i} - \widehat{Y}_i + \widehat{T}_i^T + \mu \widehat{X}_{12i}^1, \quad \widetilde{\Psi}_{44} = \mu \widehat{X}_{33i},$$

$$\widetilde{\Psi}_{14} = \widehat{P}_{1i} B_{2i} + \widehat{N}_i^T + \mu \widehat{X}_{13i}^1, \quad \widetilde{\Psi}_{33} = -\widehat{T}_i - \widehat{T}_i^T - (1-h_i)\widehat{Q} + \mu \widehat{X}_{22i},$$

$$\widetilde{\Psi}_{34} = -\widehat{N}_i^T + \mu \widehat{X}_{23i}, \quad \eta = \max_{i \in \mathscr{S}} \{|\pi_{ii}|\}, \quad V_i = B_{3i}^T \widehat{P}_{2i}, \quad U_i = A_{3i}^T \widehat{P}_{2i},$$

$$\left[\widehat{P}_{1i} \ \ \widehat{P}_{2i} \ \ \widehat{Q} \ \ \widehat{Z} \ \ \widehat{Y}_i \ \ \widehat{T}_i \ \ \widehat{N}_i \right] = \rho_{4i}^{-1} \left[P_{1i} \ \ P_{2i} \ \ Q \ \ Z \ \ Y_i \ \ T_i \ \ N_i \right],$$

$$\begin{bmatrix} \widehat{X}_{11i}^1 & \widehat{X}_{11i}^2 & \widehat{X}_{12i}^1 & \widehat{X}_{13i}^1 \\ \widehat{X}_{11i}^{2^T} & \widehat{X}_{11i}^3 & \widehat{X}_{12i}^2 & \widehat{X}_{13i}^2 \\ \widehat{X}_{12i}^{1^T} & \widehat{X}_{12i}^{2^T} & \widehat{X}_{22i} & \widehat{X}_{23i} \\ \widehat{X}_{13i}^{1^T} & \widehat{X}_{13i}^{2^T} & \widehat{X}_{23i}^T & \widehat{X}_{33i} \end{bmatrix} = \rho_{4i}^{-1} \begin{bmatrix} \widetilde{X}_{11i}^1 & \widetilde{X}_{11i}^2 & \widetilde{X}_{12i}^1 & \widetilde{X}_{13i}^1 \\ \widetilde{X}_{11i}^{2^T} & \widetilde{X}_{11i}^3 & \widetilde{X}_{12i}^2 & \widetilde{X}_{13i}^2 \\ \widetilde{X}_{12i}^{1^T} & \widetilde{X}_{12i}^{2^T} & \widetilde{X}_{22i} & \widetilde{X}_{23i} \\ \widetilde{X}_{13i}^{1^T} & \widetilde{X}_{13i}^{2^T} & \widetilde{X}_{23i}^T & \widetilde{X}_{33i} \end{bmatrix}.$$

2.5 Numerical Simulation

Example 2.1 Consider a time-delayed uncertain jump linear system (2.10) in $\mathbb{R}^2$ with two regimes $r_t \in \mathscr{S} = \{1, 2\}$. For Mode 1, the dynamics of the system are described by

$$A_{11} = \begin{bmatrix} -9 & -2 \\ 1 & -6 \end{bmatrix}, \ A_{21} = \begin{bmatrix} 2.5 & -2 \\ 2 & -1.6 \end{bmatrix}, E_{11} = \begin{bmatrix} 2.5 \\ 1 \end{bmatrix}^T, E_{21} = \begin{bmatrix} 0.4 \\ 2 \end{bmatrix}^T,$$
$$E_{41} = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}^T, B_{11} = \begin{bmatrix} 0.3 \\ 2 \end{bmatrix}, B_{21} = \begin{bmatrix} 1.5 \\ 2 \end{bmatrix}, C_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}^T, H_{11} = \begin{bmatrix} -1 \\ 2 \end{bmatrix},$$
$$E_{31} = -1, \ H_{21} = 1, \ \mu_1 = 0.1, \ h_1 = 1, \ \alpha_1 = 2.$$

For Mode 2, the dynamics of the system are described by

$$A_{12} = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}, A_{22} = \begin{bmatrix} -2 & 3 \\ 1 & -5 \end{bmatrix}, E_{12} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}^T, E_{22} = \begin{bmatrix} -0.1 \\ 1 \end{bmatrix}^T,$$
$$E_{42} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}^T, B_{12} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, B_{22} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, C_2 = \begin{bmatrix} 0.6 \\ -1 \end{bmatrix}^T, H_{12} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$
$$E_{32} = 0.3, \ H_{22} = 1, \ \mu_2 = 0.1, \ h_2 = 0.4, \ \alpha_2 = 3.$$

Let the noise attenuation level $\gamma = 1.2$, and

$$[\pi_{ij}]_{2\times 2} = \begin{bmatrix} -12 & 12 \\ 18 & -18 \end{bmatrix}, \quad [\pi_{ij}^0]_{2\times 2} = \begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix}, \quad [\pi_{ij}^1]_{2\times 2} = \begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix}.$$

Solving the LMIs in (2.39) and (2.40), we obtain

$$\widehat{P}_{111} = \begin{bmatrix} 0.645 & 0.230 \\ 0.230 & 0.553 \end{bmatrix}, \widehat{P}_{112} = \begin{bmatrix} 0.302 & -0.010 \\ -0.010 & 0.119 \end{bmatrix},$$
$$\widehat{P}_{121} = \begin{bmatrix} 0.701 & 0.266 \\ 0.266 & 0.808 \end{bmatrix}, \widehat{P}_{122} = \begin{bmatrix} 3.5454 & 1.127 \\ 1.127 & 1.604 \end{bmatrix},$$
$$\widehat{P}_{211} = \begin{bmatrix} 2.513 & 0.104 \\ 0.104 & 4.688 \end{bmatrix}, \quad \widehat{P}_{212} = \begin{bmatrix} 1.697 & -0.124 \\ -0.124 & 2.916 \end{bmatrix},$$
$$\widehat{P}_{221} = \begin{bmatrix} 1.910 & -1.141 \\ -1.141 & 5.488 \end{bmatrix}, \quad \widehat{P}_{222} = \begin{bmatrix} 3.291 & -2.969 \\ -2.969 & 12.955 \end{bmatrix},$$
$$\widehat{T}_{11} = \begin{bmatrix} 13.651 & 2.448 \\ 2.410 & 3.080 \end{bmatrix}, \widehat{T}_{12} = \begin{bmatrix} 13.228 & 1.842 \\ 1.188 & 2.868 \end{bmatrix},$$
$$\widehat{T}_{21} = \begin{bmatrix} 12.375 & 2.034 \\ 2.012 & 2.940 \end{bmatrix}, \widehat{T}_{22} = \begin{bmatrix} 12.829 & 1.838 \\ 1.741 & 2.872 \end{bmatrix},$$
$$\widehat{Y}_{11} = \begin{bmatrix} -8.435 & 6.694 \\ 5.939 & -9.778 \end{bmatrix}, \widehat{Y}_{12} = \begin{bmatrix} -13.236 & -1.843 \\ -1.168 & -2.868 \end{bmatrix},$$
$$\widehat{Y}_{21} = \begin{bmatrix} -13.028 & -2.350 \\ -3.585 & -3.703 \end{bmatrix}, \widehat{Y}_{22} = \begin{bmatrix} -12.808 & -1.838 \\ -0.044 & -2.816 \end{bmatrix},$$
$$U_{11} = \begin{bmatrix} -79.698 & 1.953 \\ 1.959 & -78.229 \end{bmatrix}, U_{12} = \begin{bmatrix} -160.118 & -5.683 \\ -5.807 & -171.548 \end{bmatrix},$$
$$U_{21} = \begin{bmatrix} -44.515 & -1.019 \\ 4.386 & -43.598 \end{bmatrix}, U_{22} = \begin{bmatrix} -120.141 & -1.413 \\ -1.525 & -101.532 \end{bmatrix},$$
$$\widehat{Q} = \begin{bmatrix} 0.0006 & 0.0002 \\ 0.0002 & 0.0034 \end{bmatrix}, \widehat{Z} = \begin{bmatrix} 1.197 & 0.176 \\ 0.176 & 0.276 \end{bmatrix}, V_{11} = \begin{bmatrix} 1.459 \\ -2.172 \end{bmatrix}^T,$$
$$V_{12} = \begin{bmatrix} -0.291 \\ -1.407 \end{bmatrix}^T, \quad V_{21} = \begin{bmatrix} 0.537 \\ -1.846 \end{bmatrix}^T, \quad V_{22} = \begin{bmatrix} 3.279 \\ -11.917 \end{bmatrix}^T,$$
$$\widehat{N}_{11} = \begin{bmatrix} 2.323 \\ 1.086 \end{bmatrix}^T, \quad \widehat{N}_{12} = \begin{bmatrix} -3.721 \\ -0.122 \end{bmatrix}^T, \quad \widehat{N}_{21} = \begin{bmatrix} 3.367 \\ 1.634 \end{bmatrix}^T,$$
$$\widehat{N}_{22} = \begin{bmatrix} -2.922 \\ -0.099 \end{bmatrix}^T, \quad K_1 = \begin{bmatrix} 4.384 \\ 1.868 \end{bmatrix}^T, \quad K_2 = \begin{bmatrix} -5.203 \\ 0.494 \end{bmatrix}^T,$$

$$\rho_{111} = 0.234, \quad \rho_{112} = 0.108, \quad \rho_{121} = 0.145, \quad \rho_{122} = 0.094,$$
$$\rho_{211} = 0.567, \quad \rho_{212} = 0.185, \quad \rho_{221} = 0.259, \quad \rho_{222} = 5.643,$$
$$\rho_{311} = 0.332, \quad \rho_{312} = 0.086, \quad \rho_{321} = 0.344, \quad \rho_{322} = 0.976,$$
$$\rho_{411} = 9.452, \quad \rho_{412} = 8.758, \quad \rho_{421} = 6.488, \quad \rho_{422} = 4.182,$$

Therefore, by Theorem 2.2, the corresponding parameters of a suitable robust output feedback control law (2.8) can be chosen as

$$A_{311} = \begin{bmatrix} -31.745 & 1.476 \\ 1.126 & -16.718 \end{bmatrix}, A_{312} = \begin{bmatrix} -94.725 & -7.757 \\ -5.994 & -59.161 \end{bmatrix},$$
$$A_{321} = \begin{bmatrix} -26.736 & -2.797 \\ -5.745 & -8.525 \end{bmatrix}, A_{322} = \begin{bmatrix} -46.141 & 9.496 \\ 10.683 & -10.013 \end{bmatrix},$$
$$B_{311} = \begin{bmatrix} -0.601 \\ -0.478 \end{bmatrix}, B_{312} = \begin{bmatrix} 0.208 \\ -0.491 \end{bmatrix}, B_{321} = \begin{bmatrix} -0.091 \\ -0.318 \end{bmatrix},$$
$$B_{322} = \begin{bmatrix} -0.209 \\ -0.872 \end{bmatrix}, K_1 = \begin{bmatrix} -4.384 \\ 1.868 \end{bmatrix}^T, K_2 = \begin{bmatrix} 5.203 \\ 0.494 \end{bmatrix}^T.$$

Example 2.2 Consider the robust stability of the uncertain system (1) with the following parameters:

$$A_{11} = \begin{bmatrix} a_{11} & 4 \\ 0 & -13 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -2.8 & -1.1 \\ 0.4 & -2 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} -0.7 & 0.2 \\ 0 & -0.1 \end{bmatrix},$$
$$A_{22} = \begin{bmatrix} 0.1 & 0.2 \\ -0.2 & -0.1 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 1.2 & 0 \\ 0 & -2.1 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0 & 1.1 \\ -4.2 & 0 \end{bmatrix},$$
$$H_{11} = \begin{bmatrix} 1.4 & 0 \\ 0 & 0.7 \end{bmatrix}, \quad H_{12} = \begin{bmatrix} 0 & 0.1 \\ -0.8 & -1.1 \end{bmatrix}, \quad E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$E_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 0.6 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0.6 & 0.3 \\ 0 & -1 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} -0.2 & 0 \\ 1 & 0 \end{bmatrix},$$
$$E_{31} = \begin{bmatrix} 0.3 & 0 \\ 0.2 & 0.1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.2 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad H_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad H_{22} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$
$$H_{31} = \begin{bmatrix} 0.1 & 0.4 \\ 0 & 1.4 \end{bmatrix}, \quad H_{32} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad [\pi_{ij}]_{2\times 2} = \begin{bmatrix} -3 & 3 \\ 5 & -5 \end{bmatrix}.$$

Table 2.1 The maximum allowed value of time delay (μ)

h		0	0.2	0.5	1.0
$a_{11} = -2$	E.K.Boukas(2002)	0.2453	0.1522	-	-
	Theorem 3.1	0.6225	0.5795	0.4930	0.3281
$a_{11} = -8$	E.K.Boukas(2002)	1.0061	0.9421	0.5834	-
	Theorem 3.1	1.2954	1.0594	0.7242	0.3427

To compare with Theorem 9.18 in [5], Theorem 2.1 should be reduced to the conditions that the jumping parameter process can be directly and precisely measured and controller can be accurately implemented. Furthermore, we also assume that $h_1 = h_2 = h$, and

$$\Delta_{A_2}(r_t, t) = H_3(r_t)F(r_t, t)E_2(r_t),$$

$$\Delta_C(r_t, t) = H_2(r_t)F(r_t, t)E_1(r_t).$$

The corresponding results are similar to Corollary 2.1, and are omitted here. The maximum allowed value of time delay for different h obtained from Theorem 2.1 are shown in Table 2.1. For comparison, The table also lists the results obtained from Theorem 9.18 in [5]. From the example, we can find that our results show much less conservatism than those in [5], especially for the increasing of the value of h.

2.6 Summary

The problem of robust output feedback H_∞ control for time-delayed uncertain jump linear systems has been studied. We have presented sufficient conditions on the existence of output feedback control by the imperfect information r_t^o, which guarantees not only the robust exponential mean-square stability but also the γ-disturbance H_∞ attenuation for the closed loop system for all admissible parameter uncertainties and time delays. However, all of these results are established under conditions of the prior knowledge of the upper bounds of the system uncertainties. A possible direction for future work is to obtain adaptive H_∞ control laws with less knowledge of those bounds.

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Chapter 3
System with Imprecise Jumping Parameters

This chapter investigates Markovian jump systems with imprecise jumping parameters. Two switching cases are considered. For asynchronous switching, a class of hybrid stochastic retarded systems with an asynchronous switching controller is studied, where the controller design relies on the observed jumping parameters that are however delayed and thus can not be measured in real-time precisely. For this case, we assume that the delayed time interval, referred to as the "asynchronous switching interval", is Markovian. The sufficient conditions under which the system is either stochastically asymptotic stable or input-to-state stable are obtained by applying the extended Razumikhin-type theorem to the asynchronous switching interval. For extended asynchronous switching, a class of switched stochastic nonlinear retarded systems in the presence of both detection delay and false alarm is studied, which are described by two independent and exponentially distributed stochastic processes, and further simplified as Markovian. Also based on the Razumikhin-type theorem incorporated with the average dwell time approach, the sufficient criteria for global asymptotic stability in probability and stochastic input-to-state stability are obtained.

3.1 Introduction

For switched systems, mode-dependent controller has received more and more attention, which is believed to be less conservative. The mode-dependent controller design for switched systems is often assumed to be strictly synchronized [5, 13, 14, 26, 27, 29, 30], which may not generally hold in reality due to unknown and unpredictable issues such as time-delay, disturbance, component and interconnection failures, etc. Specifically, in practical systems, time-delay often appears in switched systems either in input control or in output measurements, due to the distance between the place where control signal is generated and the place where control signal is applied to the plant as well as significant communication distance between the sensor and the

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controller. On the other hand, for the mode-dependent controller design, the switching information is necessary. However, due to the existence of environmental noises, disturbances, and small modelling uncertainties, considerable time is needed in the mode detection of the plant.

It thus presents a great challenge at the boundary of switched systems and time delay systems, and the concept of asynchronous switching is proposed to deal with this phenomenon. Roughly speaking, the so-called "asynchronous switching" is caused by the detection delay of switching signal which results in the mismatched period of designed controller in each subsystem. The subsystems may be unstable between these mismatched periods. Furthermore, in reality, because of the uncertainties mentioned above, false alarm (or detection error) is inevitable, which fails existing results for asynchronous switching with only detection delays. So a class of new asynchronous switching system with simultaneously considering the detection delays and the false alarms is studied. To distinguish it from the conventional asynchronous switching system, it is named the extended asynchronous switching system. Compared to the conventional asynchronous switching, the developed extended asynchronous switching can better reflect the actual situation in practical switched system control.

For conventional asynchronous switching, considerable studies have been reported, for example, state feedback stabilization [19], input-to-state stabilization [21], and output feedback stabilization [12], the use of the average dwell time approach [9, 17, 18, 24, 25], just to name a few. However, almost all the researches on asynchronous switching systems are for deterministic switched systems while the asynchronous randomly switched systems have received little attention, especially for nonlinear systems. Two difficulties are introduced in the analysis of the systems stability because of the switching signal's stochastic properties. One is that since the switching signal is a stochastic process, the methods in deterministic switched systems, e.g., dwell time approach or average dwell time approach, are difficult to be used directly; The other one is that the detected switching signal is still a stochastic process. The relationship between the detected switching signal and the origin switching signal further increases the complexity of the problem. Recently, the asynchronous issues of MJLSs have also been studied [4, 7, 22]. Among them, [22] and [7] investigated the stability and stabilization problem for a class of discrete-time MJLSs via time-delayed controller. In [4], by defining two Markov processes, the stability of the continuous-time MJLSs with detection delays and false alarms in detected switching signal and discrete-time MJLSs with constant time delays or random communication delays in mode signal are developed. Surprisingly, the studies on the stability analysis for asynchronous stochastic nonlinear systems with Markovian switching are scarce.

For extended asynchronous switching system, it has shown in [3] that the non-zero detection delay can make a closed-loop system unstable. Therefore, the existence of false alarm will inevitably further decrease the control performance. Thus, the so-called extended asynchronous switching justifies its importance. However, the coupled relationship between the true switching signal and the random detection as well as the the false alarm also increase the complexity and difficulty of stability analysis for such system. Moreover, to date switched stochastic nonlinear retarded systems (SSNLRS) under extended asynchronous switching have received little attention. All those motivate this chapter's study.

This chapter is organized as follows. In Sect. 3.2, based on a class of stochastic nonlinear systems, the formulation of asynchronous switching and extended asynchronous switching and some necessary preliminaries are stated. The global asymptotic stability and input-to-state stability are then discussed in Sect. 3.3.1. Then, the main results are extended to a class of hybrid stochastic delay systems and the simulation results are given in Sect. 3.4.1. Similar stability analysis but for the SSNLRS under extended asynchronous switching is discussed in Sect. 3.3.2, with an example given in Sect. 3.4.2. Section 3.5 concludes the chapter.

3.2 Asynchronous and Extended Asynchronous Switching

Consider the following stochastic nonlinear systems:

$$\begin{cases} \mathrm{d}x(t) = f(t, x_t, \nu(t), r(t))\mathrm{d}t + g(t, x_t, \nu(t), r(t))\mathrm{d}B(t), \\ \nu(t) = h(t, x_t, u(t), r'(t)), \end{cases} \tag{3.1}$$

with the initial state $x_0 = \{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in \mathscr{C}^b_{\mathscr{F}_0}([-\tau, 0]; \mathbb{R}^n)$ and $r_0 = r(0) = i_0$, where $x_t = \{x(t+\theta) : -\tau \leq \theta \leq 0\}$ is a $\mathscr{C}([-\tau, 0]; \mathbb{R}^n)$-valued random variable. $B(t) = (B_1(t), B_2(t), \ldots, B_m(t))^T$ is a m-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$, with Ω being the sample space, $\mathscr{F}$ being a σ-algebra, $\{\mathscr{F}_t\}_{t\geq 0}$ being a filtration and satisfies the usual conditions and P being a complete probability measure. $r(t)$ is the true switching signal and $r'(t)$ is the detected switching signal which satisfied Assumption 3.1.

In addition, in system (3.1), $\nu(t) \in \mathscr{L}^l_\infty$ is the control input, which relies on the detected switching signal $r'(t)$. $\mathscr{L}^l_\infty$ denotes the set of all the measurable and locally essentially bounded input $\nu(t) \in \mathbb{R}^l$ on $[0, \infty)$ with the norm

$$\begin{cases} \|\nu(s)\| = \inf\limits_{\mathscr{A}\subset\Omega, P(\mathscr{A})=0} \sup\{|\nu(\omega, s)| : \omega \in \Omega\backslash\mathscr{A}\} \\ \|\nu(s)\|_{[t_0,\infty)} = \sup\limits_{s\in[t_0,\infty)} \|\nu(s)\| \end{cases} \tag{3.2}$$

$u(t) \in \mathscr{L}^k_\infty$ is the reference input. Moreover, $f : \mathbb{R}_+ \times \mathscr{C}([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}^l \times \mathscr{S} \to \mathbb{R}^n$ and $g : \mathbb{R}_+ \times \mathscr{C}([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}^l \times \mathscr{S} \to \mathbb{R}^{n\times m}$ are continuous with respect to $t, x(t), u(t)$, and satisfy uniformly locally Lipschitz condition with respect to $x(t), u(t)$, and for any $i \in \mathscr{S}$, $f(t, 0, 0, i) \equiv 0$, $g(t, 0, 0, i) \equiv 0$.

Given that the true switching signal is not available for the controller design in practical, in what follows, we are concerned with the stability analysis of systems (3.1) under the following state feedback control law,

$$\nu = h(t, x_t, u, r'). \tag{3.3}$$

where $r' = r'(t)$ is the detected switching signal, $u = u(t) \in \mathscr{L}_\infty^k$ is the reference input, and $h : \mathbb{R}_+ \times \mathscr{C}([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}^k \times \mathscr{S} \to \mathbb{R}^l$ is measurable function with $h(t, 0, 0, i) \equiv 0$, for any $i \in \mathscr{S}$.

For convenience, denote

$$\bar{f}(t, x_t, u, r, r') = f(t, x_t, h(t, x_t, u, r'), r)$$
$$\bar{g}(t, x_t, u, r, r') = g(t, x_t, h(t, x_t, u, r'), r)$$

For convenience, let $\bar{f}_{ij}(t, x_t, u(t))$ and $\bar{g}_{ij}(t, x_t, u(t))$ denote $\bar{f}(t, x_t, u(t), i, j)$ and $\bar{g}(t, x_t, u(t), i, j)$, respectively, for any $i, j \in \mathscr{S}$. Specifically, when $i = j$, the mode-dependent controller and the system operate synchronously, while when $i \neq j$, they operate asynchronously. Due to $\nu(t)$ relies not on $r(t)$ but on $r'(t)$, when $r'(t) \neq r(t)$, i.e., on the asynchronous time interval, the designed controller is an mismatched one for the controlled system, which may cause the degradation of control loop performance and even make it unstable.

In the chapter, it is also assumed that $\bar{f}$, $\bar{g}$ satisfy the local Lipschitz condition and the linear growth condition , hence for the closed-loop system

$$\mathrm{d}x(t) = \bar{f}(t, x_t, u(t), r(t), r'(t))\mathrm{d}t + \bar{g}(t, x_t, u(t), r(t), r'(t))\mathrm{d}B(t) \tag{3.4}$$

there exists an unique solution on $t \geq -\tau$.

Assumption 3.1 ([11]) The values of $r(t)$ and $r'(t)$ can be divided into two cases: the quiescent case $r(t) = r'(t) = i$ and the transient case $r(t) = i, r'(t) = j, j \neq i$. In the first case, only the true modes switches and false alarms may occur. The later case corresponds to the detection delay or to the recovery from a false alarm. The only possible switch is thus a switch of $r'(t)$ from j to i, corresponding to the end of the transient, and this switch occurs on the average after $\frac{1}{\pi_{ji}^0}$ seconds. In mathematic,

Case I. When $r(t)$ has switched from i to j, $r'(t)$ follows with a delay d that is an independent exponentially distributed random variable with mean $\frac{1}{\pi_{ij}^0}$.

This is written as

$$\begin{aligned} &P\{r'(t + \Delta) = j | r'(s) = i, s \in [t^*, t], r(t^*) = j, r(t^{*-}) = i\} \\ &= \begin{cases} \pi_{ij}^0 \Delta + o(\Delta), \ i \neq j \\ 1 + \pi_{ii}^0 \Delta + o(\Delta), \ i = j. \end{cases} \end{aligned} \tag{3.5}$$

The entries of the matrix, $\varPi^0 = [\pi^0_{ij}]_{N\times N} \in \mathbb{R}^{N\times N}$, are evaluated from observed sample paths, and

$$\pi^0_{ii} = -\sum_{j\neq i} \pi^0_{ij}, \ (\pi^0_{ij} \geq 0, i \neq j). \tag{3.6}$$

Case II. When $r(t)$ remains at i, $r'(t)$ has transitioned from i to j occasionally. An independent exponential distribution with rate π^1_{ij} is again assumed

$$\begin{aligned} &P\{r'(t+\varDelta) = j|\sigma'(s) = i, s \in [t^*, t]\} \\ &= \begin{cases} \pi^1_{ij}\varDelta + o(\varDelta), \ i \neq j \\ 1 + \pi^1_{ii}\varDelta + o(\varDelta), \ i = j \end{cases} \end{aligned} \tag{3.7}$$

with a matrix, $\varPi^1 = [\pi^1_{ij}]_{N\times N} \in \mathbb{R}^{N\times N}$, of false alarm rates, which can also be valued from observed sample paths, and

$$\pi^1_{ii} = -\sum_{j\neq i} \pi^1_{ij}, \ (\pi^1_{ij} \geq 0, i \neq j). \tag{3.8}$$

According to [11], it then follows from Assumption 3.1 that:

Property 3.1 *According to Assumption 3.1, the greater π^0_{ij} is the faster detection response speed is, and the smaller π^1_{ij} is the less of the number of false alarms is, where $i, j \in \mathscr{S}$. When $\pi^0_{ij} \to \infty$ and $\pi^1_{ij} = 0$, the detection for the actual switching signal is perfect.*

3.2.1 Asynchronous Switching

The systems (3.1) under asynchronous switching are called hybrid stochastic retarded systems (HSRSs). In asynchronous switching systems, we only consider the detection delay, and ignore the detection error. According to Property 3.1, in asynchronous switching systems, $r'(t)$ should be satisfy Case I of Assumption 3.1, i.e. $\pi^1_{ij} = 0$ and $\pi^0_{ij} < \infty$. Besides, the $r(t)$ in asynchronous switching systems we consider in this chapter is Markovian, i.e. $r(t)$ is a right-continuous Markov process on the probability space taking values in a finite state space $\mathscr{S} = \{1, 2, \ldots, N\}$ with generator $\varPi = \{\pi_{ij}\}_{N\times N}$ given by

$$P\{r(t+\varDelta) = j \mid r(t) = i\} = \begin{cases} \pi_{ij}\varDelta + o(\varDelta), & i \neq j \\ 1 + \pi_{ii}\varDelta + o(\varDelta), & i = j \end{cases} \tag{3.9}$$

where $\varDelta > 0$ is a sufficiently small positive number, and $\lim_{\varDelta\to 0} \frac{o(\varDelta)}{\varDelta} = 0$. $\pi_{ij} \geq 0$ is the transition rate from i to j $(j \neq i)$, and $\pi_{ii} = -\sum_{j=1, j\neq i}^{N} \pi_{ij}$. Let $\bar{\pi} \triangleq$

$\max_{i\in\mathcal{S}}\{|\pi_{ii}|\}$ and $\tilde{\pi} \triangleq \max_{i,j\in\mathcal{S}}\{\pi_{ij}\}$ and assume the Markov process $r(t)$ is independent of the Brownian motion $B(t)$.

In the next, we make some definitions for the Markov process $r(t)$ and the detected switching signal $r'(t)$. Firstly, $r(t)$ is assumed to be a regular Markov process with standard transition probability matrix. Let the sequence $\{t_l\}_{l\geq 0}$ denote the switching instants sequence of $r(t)$, and $r(t_l) = i_l$, $t_0 = 0$. When $i_l = i$, $t_{l+1} - t_l$ is called the sojourn-time of Markov process in mode i. As usual, the sojourn-time sequence $\{t_{l+1} - t_l\}_{l\geq 0}$ belongs to an exponential distribution with rate parameter $\lambda(i)$, where $0 \leq \lambda(i) < \infty$ is the transition rate of $r(t)$ in mode i. Further, for all $i, j \in \mathcal{S}$ and $i \neq j$, $E\{t_{l+1} - t_l | i_l = i, i_{l+1} = j\} = \frac{1}{\lambda(i)}$, where $\lambda(i)$ denotes the reciprocal of the average sojourn-time of Markov process $r(t)$ in mode i. According to (3.9), we also have $\lambda(i) = -\pi_{ii}$. On the other hand, the detected switching $r'(t)$ is considered as $r'(t) = r(t - d(t))$, and it is the only switching signal which can be obtained and used by the controller. Let $\{t'_l\}_{l\geq 0}$ denote the switching instants sequence of $r'(t)$. As in [11], the following statements are assumed to describe the characteristic of $r'(t)$. When $r(t)$ jumps from i to j, $r'(t)$ follows $r(t)$ with a delay and satisfies Case I of Assumption 3.1.

Clearly, when letting $\pi^0_{ij} \to \infty$, the detection is instantaneous. It is assumed that π^0_{ij} is sufficiently large and $0 \leq d(t) \leq d \leq \inf\{t_{l+1} - t_l\}$. Further, $r'(t)$ is causal, meaning that the ordering of the switching instants of $r'(t)$ is the same as the ordering of the corresponding switching instants of $r(t)$. Thus, it follows that $0 = t_0 = t'_0 < t_1 \leq t'_1 < t_2 \leq t'_2 < \cdots < t_l \leq t'_l < t_{l+1} < \cdots$, where $t'_l = t_l + d(t_l)$ for any $l \geq 1$. Define a virtual switching signal $\bar{r}(t)$, from $[0, \infty)$ to $\mathcal{S} \times \mathcal{S}$, by $\bar{r}(t) = (r(t), r'(t))$. Let $\{\bar{t}_l\}_{l\geq 0}$ denote the switching instants of $\bar{r}(t)$. Then, for any $l \geq 1$, $\bar{t}_0 = t'_0 = t_0$, $\bar{t}_{2l-1} = t_l$ and $\bar{t}_{2l} = t'_l$.

Remark 3.1 Various algorithms exist for the detection of Markovian switching signal. In this chapter, we choose the method discussed in [11], referred to as the optimal minimum probability of error bayesian detector. As in [11], $r'(t)$ is assumed to have the similar characteristics as $r(t)$, and hence, $r'(t)$ is regarded as a conditional Markov process.

In Sect. 3.3.1, we focus on the stability analysis of system (3.1) under asynchronous switching. In system (3.1), each subsystem is described by a stochastic functional differential equation, and the switching rule between those subsystems is a continuous-time Markov process. We will consider the asynchronous case with random detection delay and model the detected switching signal as a Markov process conditional on the real Markovian switching signal. The Razumikhin-type sufficient criteria for globally asymptotically stability in probability (GASiP) [8], α-globally asymptotically stability in the mean (α-GASiM) [28], pth moment exponentially stability [10], stochastic input-to-state stability (SISS) [8], α-input-to-state stability in the mean (α-ISSiM) [28] and pth moment input-to-state stability (pth moment ISS) [2] are given. It is shown that, the stability of HSRSs under asynchronous switching can be guaranteed provided that the mode transition rate is sufficiently small, i.e., a larger instability margin can be compensated for by a smaller transition rate.

To prove these results, the following lemma is required.

Lemma 3.1 *For any given $V(x(t), t, r(t), r'(t)) \in \mathscr{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathscr{S} \times \mathscr{S}; \mathbb{R}_+)$, associated with system (3.4), the diffusion operator $\mathfrak{L}V$, from $\mathscr{C}([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+ \times \mathscr{S} \times \mathscr{S}$ to $\mathbb{R}$, can be described as follows.*

***Case I**. When $r'(t) = r(t) = i$, then*

$$\begin{aligned}
&\mathfrak{L}V(x_t, t, i, i)\\
&= V_t(x(t), t, i, i) + V_x(x(t), t, i, i)\bar{f}_{ii}(t, x_t, u)\\
&+ \frac{1}{2}tr[\bar{g}_{ii}^T(t, x_t, u)V_{xx}(x(t), t, i, i)\bar{g}_{ii}(t, x_t, u)]\\
&+ \sum_{k=1}^{N} \pi_{ik} V(x(t), t, k, i).
\end{aligned} \tag{3.10}$$

***Case II**. When $r'(t) = i$, $r(t) = j$ and $j \neq i$, then*

$$\begin{aligned}
&\mathfrak{L}V(x_t, t, j, i)\\
&= V_t(x(t), t, j, i) + V_x(x(t), t, j, i)\bar{f}_{ji}(t, x_t, u)\\
&+ \frac{1}{2}tr[\bar{g}_{ji}^T(t, x_t, u)V_{xx}(x(t), t, j, i)\bar{g}_{ji}(t, x_t, u)]\\
&+ \pi_{ij}^0 V(x(t), t, j, j) - \pi_{ij}^0 V(x(t), t, j, i).
\end{aligned} \tag{3.11}$$

Remark 3.2 Lemma 3.1 is from (2) in [2] and Lemma 3 in [4]. When $r'(t) \equiv r(t)$ for all $t \geq 0$, (3.10) is the same as (2) in [2]. Otherwise, (3.10) and (3.11) are similar to the ones in Lemma 3 in [4]. Lemma 3 in [4] considers also false alarms of $r'(t)$. In asynchronous switching systems, the causality of $r'(t)$ means $\Pi^1 = \{\pi_{ij}^1\}_{N\times N} = 0$ and (3.10) follows.

3.2.2 *Extended Asynchronous Switching*

The systems (3.1) under extended asynchronous switching are called switched stochastic nonlinear retarded systems (SSNLRS). Note that, for extended asynchronous switching systems, the $r(t)$ considered in system (3.1) is deterministic.

In this scenario, $r = r(t) : [t_0, \infty) \to \mathscr{S}$ ($\mathscr{S}$ is the index set, and may be infinite) is the switching law and is right hand continuous and piecewise constant on t, $r(t)$ discussed in extended asynchronous switching systems is time dependent, and the corresponding switching instants sequence is $\{t_l\}_{l\geq 0}$. The i_lth subsystems will be activated at time interval $[t_l, t_{l+1})$. Specially, when $t = t_0$ (t_0 is the initial time), suppose $r_0 = r(t_0) = i_0 \in \mathscr{S}$. Besides, $r'(t)$ is the the detected switching signal satisfied Assumption 3.1.

From Assumption 3.1, under any time interval $[t_m, t_{m+1})$, where $t_m, t_{m+1} \in \{t_l\}_{l\geq 0}$, the number of switches of $r'(t)$ can only be the following two cases: $2k + 1$ and

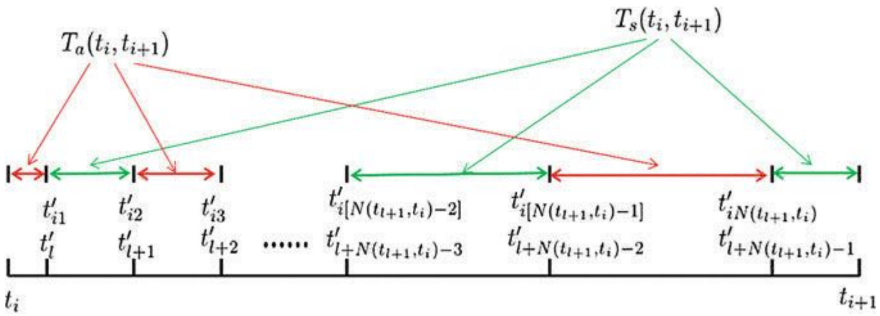


Fig. 3.1 The re-definition of $\{t'_l\}_{l\geq 0}$ on interval $[t_i, t_{i+1})$

$2k$, where $k \geq 0$ is the switch number of $r'(t)$ which caused by false alarm. We assume $r(t_m) = i_m$. First, $r'(t)$ will first switch to i_m with responding to transient case, i.e., detection delay process, and the detection delay doesn't equal to zero. After $r'(t) = i_m, t \in (t_m, t_{m+1})$, if a false alarm occurs, the next switch is that $r'(t)$ switch to i_m (recovery from the false alarm mode). Thus, before time t_{m+1}, the total switch number of $r'(t)$ is $2k + 1$. Second, if the detection delay is zero, i.e., $r'(t_m) = r(t_m) = i_m$, then total switch number of $r'(t)$ on $[t_m, t_{m+1})$ will be $2k$.

Let $\{t'_l\}_{l\geq 0}$ denote the switching instants sequence of $r'(t)$, with $t_0 = t'_0$ and $r'(t'_0) = r(t_0)$. For any $i \in \mathbb{N}_+ \cup \{0\}$, let $N(t_{i+1}, t_i)$ denote the number of switches of $r'(t)$ on $[t_i, t_{i+1})$. Moreover, as shown in Fig. 3.1, we subdivide the sequence $\{t'_l\}_{l\geq 0}$ into a sequence of subsets, i.e., $\{t'_l\}_{l\geq 0} = \bigcup_i \{t'_{i1}, t'_{i2}, \ldots, t'_{iN(t_{i+1},t_i)}\}$, such that $\{t'_{i1}, t'_{i2}, \ldots, t'_{iN(t_{i+1},t_i)}\} \subset [t_i, t_{i+1})$. In the sequel, we assume that $r(t_i^-) = r'(t_i^-)$ and $r'(t) = r(t) = r(t_0) = i_0$, for any $t \in (t_0, t_1)$. Note that, for any $i \in \mathbb{N}_+ \cup \{0\}$, $r(t_i^-) = r'(t_i^-)$ means that the switches between the subsystems of switched system occur in the case that the controller and the system operate synchronously. The hypothesis is commonly employed in the context in asynchronous switching systems, in which there always exists the period that the controller and the system run synchronously [16–18, 24, 25].

For any $i \in \mathbb{N}$, if the detection delay is non-zero, then the controller mode is strictly synchronous with the system on the following time intervals: $[t'_{i1}, t'_{i2})$, $[t'_{i3}, t'_{i4}), \ldots, [t'_{iN(t_{i+1},t_i)}, t_{i+1})$.

We define $T_s(t_0, t_1) = [t_0, t_1)$, $T_s(t_i, t_{i+1}) = \bigcup_{j=1,3,\ldots,N(t_{i+1},t_i)} [t'_{ij}, t'_{i(j+1)})$, and for simplicity $T_a(t_i, t_{i+1}) = \bigcup_{j=0,2,\ldots,N(t_{i+1},t_i)-1} [t'_{ij}, t'_{i(j+1)})$, where $t'_{i(N(t_{i+1},t_i)+1)} = t_{i+1}$, $t'_{i0} = t_i$. However, if the detection delay is equal to zero, then the controller mode is strictly synchronous with the system on the following time intervals: $[t_i, t'_{i1})$, $[t'_{i2}, t'_{i3}), \ldots, [t'_{iN(t_{i+1},t_i)}, t_{i+1})$. In this case, $T_s(t_i, t_{i+1}) = \bigcup_{j=0,2,\ldots,N(t_{i+1},t_i)} [t'_{ij}, t'_{i(j+1)})$, and $T_a(t_i, t_{i+1}) = \bigcup_{j=1,3,\ldots,N(t_{i+1},t_i)-1} [t'_{ij}, t'_{i(j+1)})$. Then, $T_s(t_i, t_{i+1}) \bigcap T_a(t_i, t_{i+1}) = \varnothing$, $[t_i, t_{i+1}) = T_a(t_i, t_{i+1}) \bigcup T_s(t_i, t_{i+1})$. In the sequel, we let $T_a(t - s)$ denote the length of $T_a(t, s)$, for any $t \geq s \geq t_0$.

To simplify the expression, the next definition is needed.

Definition 3.1 [6] For any given constants $\tau^* > 0$ and N_0, let $N_r(t, s)$ denote the switch number of $r(t)$ in $[s, t)$, for any $t > s \geq t_0$, and let

$$S[\tau^*, N_0] = \{r(\cdot):\ N_r(t,s) \le N_0 + \frac{t-s}{\tau^*},\ \forall s \in [t_0, t)\}.$$

then τ^* is called the average dwell-time of $S[\tau^*, N_0]$, and $\tau_r \triangleq \sup_{t\ge t_0} \sup_{t>s\ge t_0} \frac{t-s}{N_r(t,s)-N_0}$ is called the average dwell-time of $r(\cdot)$.

In Sect. 3.3.2, stochastic input-to-state stability for system (3.1) under extended asynchronous switching will be investigated. The Razumikhin-type stability criteria based on average dwell time approach are developed for the proposed extended asynchronous switching system.

3.3 Stability Analysis Under the Two Switchings

3.3.1 Stability Analysis Under Asynchronous Switching

From the definition of ISS, an ISS system is GAS if the input $u \equiv 0$. Therefore, the GAS property is useful for ISS. In this section, GAS in probability and in pth moment are considered.

To begin with, a useful lemma is stated as follows.

Lemma 3.2 *Let* $V(t) = e^{\lambda t} V(x(t), t, \bar{r}(t)) = e^{\lambda t} V(x(t), t, r(t), r'(t))$ *for all* $t \ge 0$ *and* $\lambda \ge 0$*, then*

$$\begin{aligned} &D^+ E\{V(t)\} = E\{\mathfrak{L}V(t)\} \\ &= \lambda E\{V(t)\} + e^{\lambda t} E\{\mathfrak{L}V(x_t, t, r(t), r'(t)))\}, \end{aligned} \tag{3.12}$$

where $D^+ E\{V(t)\} = \limsup_{dt\to 0^+} \frac{E\{V(t+dt)\}-E\{V(t)\}}{dt}$.

Proof Firstly, for any $k_1, k_2 \in \mathscr{S}$, it follows

$$\begin{aligned} &E\{V(t+dt)|x(t), r(t) = k_1, r'(t) = k_2, t\} \\ &= E\{V(t) + \lambda V(t)dt|x(t), r(t) = k_1, r'(t) = k_2, t\} \\ &+ E\{e^{\lambda t} V_t(x(t), t, \bar{r}(t))dt|x(t), r(t) = k_1, r'(t) = k_2, t\} \\ &+ E\{e^{\lambda t} V_x(x(t), t, \bar{r}(t)) \bar{f}(t, x_t, u, \bar{r}(t))dt \\ &+ \frac{1}{2} e^{\lambda t} tr[\bar{g}^T(t, x_t, u, \bar{r}(t)) V_{xx}(x(t), t, \bar{r}(t)) \\ &\times \bar{g}(t, x_t, u, \bar{r}(t))]dt|x(t), r(t) = k_1, r'(t) = k_2, t\} \end{aligned}$$

$$\begin{aligned}&+ E\{e^{\lambda t}V(x(t), t, r(t+dt), r'(t))\\&+ e^{\lambda t}V(x(t), t, r(t), r'(t+dt))|x(t), r(t) = k_1,\\&r'(t) = k_2, t\} + o(dt),\end{aligned}\tag{3.13}$$

which is in accordance with Lemma 3.1. We complete the proof by considering the following two cases: $r(t) = r'(t) = i$ and $r'(t) = i$, $r(t) = j$, respectively, where $i, j \in \mathscr{S}$ and $j \neq i$.

Case I. $r'(t) = r(t) = i$.

In this case, only the true mode switches may occur. Using the conclusion in [11], it follows

$$\begin{aligned}&E\{e^{\lambda t}V(x(t), t, r(t+dt), r'(t))|x(t), r(t) = r'(t) = i, t\}\\&= \sum_{j=1}^{N} \pi_{ij}[e^{\lambda t}V(x(t), t, j, i) - e^{\lambda t}V(x(t), t, i, i)]dt\\&= \sum_{j=1}^{N} \pi_{ij}e^{\lambda t}V(x(t), t, j, i)dt,\\&E\{e^{\lambda t}V(x(t), t, r(t), r'(t+dt))|x(t), r(t) = r'(t) = i, t\}\\&= \pi_{ii}^{1}[e^{\lambda t}V(x(t), t, i, i) - e^{\lambda t}V(x(t), t, i, i)]dt = 0.\end{aligned}$$

Then,

$$\begin{aligned}&E\{V(t+dt)|x(t), r'(t) = r(t) = i, t\}\\&= E\{V(t)|x(t), r'(t) = r(t) = i, t\}\\&+ [\lambda e^{\lambda t}V(x(t), t, i, i) + e^{\lambda t}\mathfrak{L}V(x_t, t, i, i)]dt + o(dt),\end{aligned}\tag{3.14}$$

where $\mathfrak{L}V(x_t, t, i, i)$ is defined in (3.10). Taking the expectation on the both sides of (3.14),

$$\mathrm{D}^{+}E\{e^{\lambda t}V(x(t), t, i, i)\} = E\{\lambda e^{\lambda t}V(x(t), t, i, i) + e^{\lambda t}\mathfrak{L}V(x_t, t, i, i)\}.\tag{3.15}$$

Case II. $r'(t) = i, r(t) = j$.

This situation corresponds to the detection delay, and it is assumed that the true mode $r(t)$ doesn't switch during this short time lapse. The only possible switch is that $r'(t)$ switches from i to j, corresponding to the end of the transient, and this switch occurs on the average after $\frac{1}{\pi_{ij}^{0}}$ seconds.

Then,

$$E\{e^{\lambda t}V(x(t),t,r(t+dt),r'(t))|_{\substack{x(t),r(t)=j\\ r'(t)=i,t}}\}$$
$$=\pi_{jj}[e^{\lambda t}V(x(t),t,j,i)-e^{\lambda t}V(x(t),t,j,i)]dt=0,$$
$$E\{e^{\lambda t}V(x(t),t,r(t),r'(t+dt))|_{\substack{x(t),r(t)=j\\ r'(t)=i,t}}\}$$
$$=\pi^0_{ij}[e^{\lambda t}V(x(t),t,j,j)-e^{\lambda t}V(x(t),t,j,i)]dt.$$

Thus, similar to (3.14), it holds that

$$\begin{aligned}&\mathrm{D}^+E\{e^{\lambda t}V(x(t),t,j,i)\}\\&=E\{\lambda e^{\lambda t}V(x(t),t,j,i)+e^{\lambda t}\mathfrak{L}V(x_t,t,j,i)\},\end{aligned} \tag{3.16}$$

where $\mathfrak{L}V(x_t,t,j,i)$ in this case is defined in (3.11).

Combining (3.15) and (3.16), and considering the arbitrary of i, j, it follows (3.12), for $t\geq 0$. Thus we complete the proof.

Using Lemma 3.2, the criteria of GASiP for system (3.4) is obtained.

Theorem 3.1 *System (3.4) with* $u\equiv 0$ *is GASiP if there exist functions* $\alpha_1\in\mathscr{K}_\infty$, $\alpha_2\in\mathscr{CK}_\infty$, *constants* $\mu\geq 1$, $q>1$, λ_2, $0<\varsigma<1$, *and* $V(x(t),t,\bar{r}(t))\in C^{2,1}(\mathbb{R}^n\times\mathbb{R}_+\times\mathscr{S}\times\mathscr{S};\mathbb{R}_+)$, *such that*

$$\alpha_1(|x(t)|)\leq V(x(t),t,\bar{r}(t))\leq\alpha_2(|x(t)|) \tag{3.17}$$

and for any $l\in\mathbb{N}_+$, *there exists* $\bar{\lambda}_1\in(0,\lambda_1)$ *such that*

$$\begin{aligned}&E\{\mathfrak{L}V(\varphi(\theta),t,\bar{r}(t))\}\\&\leq\begin{cases}-\lambda_1E\{V(\varphi(0),t,\bar{r}(t))\},\ t\in[\bar{t}_{2l-2},\bar{t}_{2l-1})\\ \lambda_2E\{V(\varphi(0),t,\bar{r}(t))\},\quad t\in[\bar{t}_{2l-1},\bar{t}_{2l})\end{cases}\end{aligned} \tag{3.18}$$

provided those $\varphi\in L^p_{\mathscr{F}_t}([-\tau,0];\mathbb{R}^n)$ *satisfying that*

$$\min_{i,j\in\mathscr{S}}E\{V(\varphi(\theta),t+\theta,i,j)\}<qE\{V(\varphi(0),t,\bar{r}(t))\}, \tag{3.19}$$

where

$$e^{\bar{\lambda}_1\tau}<q \tag{3.20}$$

and moreover,

$$E\{V(x(\bar{t}_l),\bar{t}_l,\bar{r}(\bar{t}_l))\}\leq\mu E\{V(x(\bar{t}_l),\bar{t}_l,\bar{r}(\bar{t}_{l-1}))\} \tag{3.21}$$

with some $\bar{\lambda}_2 \in (\lambda_2, \infty)$ *such that*

$$\mu^2 e^{(\bar{\lambda}_1+\bar{\lambda}_2)d}\bar{\pi} - \tilde{\pi} \leq \varsigma\bar{\lambda}_1. \tag{3.22}$$

Proof According to (3.12) in Lemma 3.2, we have

$$\mathrm{D}^+ E\{V(x(t), t, \bar{r}(t))\} = E\{\mathfrak{L}V(x_t, t, \bar{r}(t))\}, \tag{3.23}$$

for any $t \in [\bar{t}_{2l-2}, \bar{t}_{2l-1}) \cup [\bar{t}_{2l-1}, \bar{t}_{2l})$, $l \in \mathbb{N}_+$, with $\bar{t}_0 = t_0 = t'_0 = 0$.

On the one hand, from (3.17), using Jensen's inequality, one can obtain

$$E\{V(x(t), t, i_0, i_0)\} = E\{V(x(t), t, \bar{r}(t))\} \leq E\{\alpha_2(|x(t)|)\} \leq \alpha_2(E\{\|\xi\|\}),$$

for any $t \in [t_0 - \tau, t_0]$.

In the following, we shall prove that

$$E\{V(x(t), t, i_0, i_0)\} \leq \alpha_2(E\{\|\xi\|\})e^{-\bar{\lambda}_1(t-t_0)}, \tag{3.24}$$

for $t \in [\bar{t}_0, \bar{t}_1) = [t_0, t_1)$.

Suppose (3.24) is not true, i.e., there exists some $t \in (t_0, t_1)$ such that

$$E\{V(x(t), t, i_0, i_0)\} > \alpha_2(E\{\|\xi\|\})e^{-\bar{\lambda}_1(t-t_0)}. \tag{3.25}$$

Let $t^* = \inf\{t \in (t_0, t_1) : E\{V(x(t), t, i_0, i_0)\} > \alpha_2(E\{\|\xi\|\})e^{-\bar{\lambda}_1(t-t_0)}\}$. Then $t^* \in (t_0, t_1)$ and $E\{V(x(t^*), t^*, i_0, i_0)\} = \alpha_2(E\{\|\xi\|\})e^{-\bar{\lambda}_1(t^*-t_0)}$. Further, there exists a sequence $\{\tilde{t}_n\}$ ($\tilde{t}_n \in (t^*, t_1)$, for any $n \in \mathbb{N}_+$) with $\lim_{n\to\infty} \tilde{t}_n = t^*$, such that

$$E\{V(x(\tilde{t}_n), \tilde{t}_n), i_0, i_0\} > \alpha_2(E\{\|\xi\|\})e^{-\bar{\lambda}_1(\tilde{t}_n-t_0)}. \tag{3.26}$$

From the definition of t^*, for any $\theta \in [-\tau, 0]$, it follows

$$\begin{aligned} & E\{V(x(t^*+\theta), t^*+\theta, i_0, i_0)\} \\ & \leq e^{-\bar{\lambda}_1\theta} E\{V(x(t^*), t^*, i_0, i_0)\} \\ & \leq e^{\bar{\lambda}_1\tau} E\{V(x(t^*), t^*, i_0, i_0)\}, \end{aligned}$$

and further, for $\theta \in [-\tau, 0]$,

$$\min_{i,j\in\mathscr{S}} E\{V(x(t^*+\theta), t^*+\theta, i, j)\} < qE\{V(x(t^*), t^*, i_0, i_0)\},$$

thus, from (3.18) and (3.23), we obtain

$$\begin{aligned}\mathrm{D}^{+}E\{V(x(t^*),t^*,i_0,i_0)\} &\leq -\lambda_1 E\{V(x(t^*),t^*,i_0,i_0)\}\\ &< -\bar{\lambda}_1 E\{V(x(t^*),t^*,i_0,i_0)\}.\end{aligned}$$

Then, for $h > 0$ which is sufficient small, it holds

$$\mathrm{D}^{+}E\{V(x(t^*),t^*,i_0,i_0)\} \leq -\bar{\lambda}_1 E\{V(x(t^*),t^*,i_0,i_0)\},$$

for $t \in [t^*, t^* + h]$.

Hence,

$$E\{V(x(t^*+h),t^*+h,i_0,i_0)\} \leq E\{V(x(t^*),t^*,i_0,i_0)\}e^{-\bar{\lambda}_1 h},$$

which is a contradiction to (3.26). Therefore, (3.24) holds. Combining the continuity of function $V(x(t),t,i_0,i_0)$ and (3.21), we have

$$E\{V(x(\bar{t}_1),\bar{t}_1,\bar{r}(\bar{t}_1))\} \leq \mu E\{V(x(\bar{t}_1),\bar{t}_1,\bar{r}(\bar{t}_0))\} \leq \mu\alpha_2(E\{\|\xi\|\})e^{-\bar{\lambda}_1(t_1-t_0)}. \tag{3.27}$$

Let $W(t,\bar{r}(t)) = e^{\bar{\lambda}_1 t}V(x(t),t,\bar{r}(t))$. In the sequel, we will show that for any $t \in [\bar{t}_{2l-1}, \bar{t}_{2l+1})$,

$$E\{W(t,\bar{r}(t))\} \leq \mu E\{W(\bar{t}_{2l-1},\bar{r}(\bar{t}_{2l-1}))\}e^{(\bar{\lambda}_1+\bar{\lambda}_2)d}. \tag{3.28}$$

The following three cases are considered: $t \in [\bar{t}_{2l-1}, \bar{t}_{2l})$, $t = \bar{t}_{2l}$ and $t \in (\bar{t}_{2l}, \bar{t}_{2l+1})$. First, when $t \in [\bar{t}_{2l-1}, \bar{t}_{2l})$, we claim that

$$E\{W(t,\bar{r}(t))\} \leq \mu E\{W(\bar{t}_{2l-1},\bar{r}(\bar{t}_{2l-1}))\}e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t-\bar{t}_{2l-1})}. \tag{3.29}$$

Suppose (3.29) is not true. Then, there exists some $t \in [\bar{t}_{2l-1}, \bar{t}_{2l})$ such that

$$E\{W(t,\bar{r}(t))\} > \mu E\{W(\bar{t}_{2l-1},\bar{r}(\bar{t}_{2l-1}))\}e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t-\bar{t}_{2l-1})}.$$

Let

$$\begin{aligned}t^* = \inf\{t \in [\bar{t}_{2l-1},\bar{t}_{2l}) : E\{W(t,\bar{r}(\bar{t}_{2l}))\} >\\ \mu E\{W(\bar{t}_{2l-1},\bar{r}(\bar{t}_{2l-1}))\}e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t-\bar{t}_{2l-1})}\},\end{aligned}$$

thus

$$E\{W(t^*,\bar{r}(\bar{t}_{2l}))\} = \mu E\{W(\bar{t}_{2l-1},\bar{r}(\bar{t}_{2l-1}))\}e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t^*-\bar{t}_{2l-1})}.$$

Considering the continuity, there exists a list of sequence $\{\tilde{t}_n\}_{n\in\mathbb{N}_+} \in (t^*, \bar{t}_{2l})$ with $\lim_{n\to\infty} \tilde{t}_n = t^*$ such that

$$E\{W(\tilde{t}_n, \bar{r}(\bar{t}_{2l}))\} > \mu E\{W(\bar{t}_{2l-1}, \bar{r}(\bar{t}_{2l-1}))\}e^{(\bar{\lambda}_1+\bar{\lambda}_2)(\tilde{t}_n-\bar{t}_{2l-1})}. \tag{3.30}$$

Define $U(t) = e^{-(\bar{\lambda}_1+\bar{\lambda}_2)t}E\{W(t, \bar{r}(t))\}$, then

$$\mathrm{D}^+U(t) = -\bar{\lambda}_2 e^{-\bar{\lambda}_2 t}E\{V(x(t), t, \bar{r}(t))\} + e^{-\bar{\lambda}_2 t}\mathrm{D}^+E\{V(x(t), t, \bar{r}(t))\}.$$

From the definition of t^*, for any $\theta \in [-\tau, 0]$, it follows

$$\begin{aligned}
&\mu E\{W(\bar{t}_{2l-1}, \bar{r}(\bar{t}_{2l-1}))\}e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t^*+\theta-\bar{t}_{2l-1})}\\
&= E\{W(t^*, \bar{r}(\bar{t}_{2l-1}))\}e^{(\bar{\lambda}_1+\bar{\lambda}_2)\theta}\\
&\geq E\{W(t^*+\theta, \bar{r}(\bar{t}_{2l-1}))\},
\end{aligned}$$

which means

$$\begin{aligned}
&E\{V(x(t^*+\theta), t^*+\theta, \bar{r}(\bar{t}_{2l-1}))\}\\
&\leq E\{V(x(t^*), t^*, \bar{r}(\bar{t}_{2l-1}))\}e^{\bar{\lambda}_2\theta} \leq E\{V(x(t^*), t^*, \bar{r}(\bar{t}_{2l-1}))\}.
\end{aligned} \tag{3.31}$$

Hence,

$$\min_{i,j\in\mathscr{S}} E\{V(x(t^*+\theta), t^*+\theta, i, j)\} < qE\{V(x(t^*), t^*, \bar{r}(\bar{t}_{2l-1}))\}.$$

Then,

$$\begin{aligned}
\mathrm{D}^+U(t^*) &= -\bar{\lambda}_2 e^{-\bar{\lambda}_2 t^*}E\{V(x(t^*), t^*, \bar{r}(\bar{t}_{2l-1}))\}\\
&\quad + e^{-\bar{\lambda}_2 t^*}\mathrm{D}^+E\{V(x(t^*), t^*, \bar{r}(\bar{t}_{2l-1}))\}\\
&\leq -(\bar{\lambda}_2 - \lambda_2)e^{-\bar{\lambda}_2 t^*}E\{V(x(t^*), t^*, \bar{r}(\bar{t}_{2l-1}))\}.
\end{aligned}$$

Note that either $E\{V(x(t^*), t^*, \bar{r}(\bar{t}_{2l-1}))\} = 0$ or $E\{V(x(t^*), t^*, \bar{r}(\bar{t}_{2l-1}))\} > 0$. In the case $E\{V(x(t^*), t^*, \bar{r}(\bar{t}_{2l-1}))\} = 0$, we have $x(t^*) = 0$ a.s. From (3.31) and (3.17), we have $x(t^*+\theta) = 0$ a.s. for any $\theta \in [-\tau, 0]$. Recalling that $h(t^*, 0, 0, r'(\bar{t}_{2l-1})) = 0$, $f(t^*, 0, 0, r(\bar{t}_{2l-1})) = 0$ and $g(t^*, 0, 0, r(\bar{t}_{2l-1})) = 0$, hence $\bar{f}(t^*, 0, 0, \bar{r}(\bar{t}_{2l-1})) = 0$ and $g(t^*, 0, 0, \bar{r}(\bar{t}_{2l-1})) = 0$. Thus, one sees that $x(t^*+h) = 0$ a.s., for all $h > 0$, i.e., $E\{W(t^*+h, \bar{r}(\bar{t}_{2l-1}))\} = 0$, which is a contradiction of (3.30).

On the other hand, in the case $E\{V(x(t^*), t^*, \bar{r}(\bar{t}_{2l-1}))\} > 0$, there exists a positive number h which is sufficient small such that $\mathrm{D}^+U(t) \leq 0$, for all $t \in [t^*, t^*+h]$, which means

$$E\{W(t^*+h, \bar{r}(\bar{t}_{2l-1}))\} \leq e^{(\bar{\lambda}_1+\bar{\lambda}_2)h}E\{W(t^*, \bar{r}(\bar{t}_{2l-1}))\}$$

and it is a contradiction to (3.30). Therefore (3.29) holds. Further, (3.28) holds on $t \in [\bar{t}_{2l-1}, \bar{t}_{2l})$.

By considering the continuity of $W(t, \bar{r}(\bar{t}_{2l-1}))$ at time $t = \bar{t}_{2l}$, it follows

$$E\{W(\bar{t}_{2l}, \bar{r}(\bar{t}_{2l}))\} \leq \mu E\{W(\bar{t}_{2l-1}, \bar{r}(\bar{t}_{2l-1}))\} e^{(\bar{\lambda}_1+\bar{\lambda}_2)d}.$$

Following the similar analysis on interval $(\bar{t}_{2l-1}, \bar{t}_{2l})$, one can prove that (3.28) holds on $(\bar{t}_{2l}, \bar{t}_{2l+1})$, and then it holds on $[\bar{t}_{2l-1}, \bar{t}_{2l+1})$.

Thus,

$$\begin{aligned} & E\{V(x(t), t, \bar{r}(t))\} \\ & \leq \mu E\{V(x(\bar{t}_{2l-1}), \bar{t}_{2l-1}, \bar{r}(\bar{t}_{2l-1}))\} e^{-\bar{\lambda}_1(t-\bar{t}_{2l-1})} \times e^{(\bar{\lambda}_1+\bar{\lambda}_2)d}, \end{aligned} \tag{3.32}$$

where $t \in [\bar{t}_{2l-1}, \bar{t}_{2l+1})$.

By considering the continuity of $V(x(t), t, \bar{r}(\bar{t}_{2l}))$, one can see that (3.32) holds at time $\bar{t}_{2l+1}$, and then,

$$\begin{aligned} & E\{V(x(t_{l+1}), t_{l+1}, \bar{r}(t_{l+1}))\} \\ & \leq \mu^2 E\{V(x(t_l), t_l, \bar{r}(t_l))\} e^{-\bar{\lambda}_1(t_{l+1}-t_l)} e^{(\bar{\lambda}_1+\bar{\lambda}_2)d}. \end{aligned} \tag{3.33}$$

For any $t \geq \bar{t}_1 = t_1$, iterating (3.32) from $l = 1$ to $l = N_r(t, t_1) + 1$, one can obtain

$$\begin{aligned} & E\{V(x(t), t, \bar{r}(t))\} \\ & \leq \mu^2 E\{V(x(t_{N_r(t,t_1)+1}), t_{N_r(t,t_1)+1}, \bar{r}(t_{N_r(t,t_1)+1}))\} \times e^{(\bar{\lambda}_1+\bar{\lambda}_2)d} e^{-\bar{\lambda}_1(t-t_{N_r(t,t_1)+1})} \\ & = E\{\mu^{2(N_r(t,t_1)+1-N_r(t,t_1))} e^{(N_r(t,t_1)+1-N_r(t,t_1))(\bar{\lambda}_1+\bar{\lambda}_2)d}\} \\ & \times E\{V(x(t_{N_r(t,t_1)+1}), t_{N_r(t,t_1)+1}, \bar{r}(t_{N_r(t,t_1)+1}))\} \times e^{-\bar{\lambda}_1(t-t_{N_r(t,t_1)+1})} \\ & \leq E\{\mu^{2(N_r(t,t_1)+1-N_r(t,t_1))} e^{(N_r(t,t_1)+1-N_r(t,t_1))(\bar{\lambda}_1+\bar{\lambda}_2)d}\} \\ & \times \mu^2 e^{(\bar{\lambda}_1+\bar{\lambda}_2)d} E\{V(x(t_{N_r(t,t_1)}), t_{N_r(t,t_1)}, \bar{r}(t_{N_r(t,t_1)}))\} \times e^{-\bar{\lambda}_1(t-t_{N_r(t,t_1)})} \\ & = E\{\mu^{2(N_r(t,t_1)+2-N_r(t,t_1))} e^{(N_r(t,t_1)+2-N_r(t,t_1))(\bar{\lambda}_1+\bar{\lambda}_2)d}\} \\ & \times E\{V(x(t_{N_r(t,t_1)}), t_{N_r(t,t_1)}, \bar{r}(t_{N_r(t,t_1)}))\} \times e^{-\bar{\lambda}_1(t-t_{N_r(t,t_1)})} \\ & \leq \cdots \\ & \leq E\{\mu^{2(N_r(t,t_1)-2)} e^{(N_r(t,t_1)-2)(\bar{\lambda}_1+\bar{\lambda}_2)d}\} \mu^2 e^{(\bar{\lambda}_1+\bar{\lambda}_2)d} \\ & \times E\{V(x(t_2), t_2, \bar{r}(t_2))\} e^{-\bar{\lambda}_1(t-t_2)} \\ & = E\{\mu^{2(N_r(t,t_1)-1)} e^{(N_r(t,t_1)-1)(\bar{\lambda}_1+\bar{\lambda}_2)d}\} \times E\{V(x(t_2), t_2, \bar{r}(t_2))\} e^{-\bar{\lambda}_1(t-t_2)} \\ & \leq E\{\mu^{2(N_r(t,t_1)-1)} e^{(N_r(t,t_1)-1)(\bar{\lambda}_1+\bar{\lambda}_2)d}\} \mu^2 e^{(\bar{\lambda}_1+\bar{\lambda}_2)d} \\ & \times E\{V(x(t_1), t_1, \bar{r}(t_1))\} e^{-\bar{\lambda}_1(t-t_1)} \\ & = E\{\mu^{2N_r(t,t_1)} e^{N_r(t,t_1)(\bar{\lambda}_1+\bar{\lambda}_2)d}\} \times E\{V(x(t_1), t_1, \bar{r}(t_1))\} e^{-\bar{\lambda}_1(t-t_1)}. \end{aligned} \tag{3.34}$$

Combining (3.27) with (3.34), we arrive at

$$E\{V(x(t), t, \bar{r}(t))\} \leq E\{\mu^{2N_r(t,0)} e^{(\bar{\lambda}_1+\bar{\lambda}_2)N_r(t,0)d}\} \times \alpha_2(E\{\|\xi\|\}) e^{-\bar{\lambda}_1 t}, \tag{3.35}$$

for any $t \geq t_0 - \tau$.

According to Lemma 6 in [1], let $s = 2\ln(\mu) + (\bar{\lambda}_1 + \bar{\lambda}_2)d$, there exists a positive number $M > 0$ such that

$$e^{-\varsigma\bar{\lambda}_1 t} E\{\mu^{2N_r(t,0)} e^{(\bar{\lambda}_1+\bar{\lambda}_2)N_r(t,0)d}\} \leq M e^{-\varsigma\bar{\lambda}_1 t} + e^{[\mu^2 e^{(\bar{\lambda}_1+\bar{\lambda}_2)d}\bar{\pi} - \tilde{\pi} - \varsigma\bar{\lambda}_1]t}.$$

When $\varsigma\bar{\lambda}_1 \geq \mu^2 e^{(\bar{\lambda}_1+\bar{\lambda}_2)d}\bar{\pi} - \tilde{\pi}$, we have

$$e^{-\varsigma\bar{\lambda}_1 t} E\{\mu^{2N_r(t,0)} e^{(\bar{\lambda}_1+\bar{\lambda}_2)N_r(t,0)d}\} \leq M + 1 < \infty.$$

Then,

$$E\{V(x(t), t, \bar{r}(t))\} \leq \bar{M} e^{-(1-\varsigma)\bar{\lambda}_1 t} \alpha_2(E\{\|\xi\|\}) \triangleq \bar{\beta}(E\{\|\xi\|\}, t), \tag{3.36}$$

for any $M + 1 \leq \bar{M} < \infty$.

It's no difficulty to verify $\bar{\beta}(\cdot, \cdot) \in \mathscr{K}\mathscr{L}$ when $0 < \varsigma < 1$. Then, for any $\varepsilon > 0$, take $\tilde{\beta} = \frac{\bar{\beta}}{\varepsilon}$. Obviously, $\tilde{\beta}(\cdot, \cdot) \in \mathscr{K}\mathscr{L}$. Using Chebyshev's inequality, we have

$$P\{V(x(t), t, \bar{r}(t)) \geq \tilde{\beta}(E\{\|\xi\|\}, t)\} \leq \frac{E\{V(x(t), t, \bar{r}(t))\}}{\tilde{\beta}(E\{\|\xi\|\}, t)} < \varepsilon,$$

i.e.

$$P\{|x(t)| < \beta(E\{\|\xi\|\}, t)\} \geq 1 - \varepsilon,$$

where $\beta(r, s) = \alpha_1^{-1} \circ \tilde{\beta}(r, s) \in \mathscr{K}\mathscr{L}$. Thus, we complete the proof.

Remark 3.3 (i). Assumption (3.18) is widely used in Razumikhin-type stability criterion and imposes less restrictions on the functions $\bar{f}(t, \varphi(\theta), u(t), \bar{r}(t))$ and $\bar{g}(t, \varphi(\theta), u(t), \bar{r}(t))$, as described in [10]. When $t \in [\bar{t}_{2l-1}, \bar{t}_{2l})$, condition (3.18) corresponds to the asynchronous case and λ_2 may or may not be positive. In what follows, λ_2 is assumed to positive, and λ_1 and λ_2 denote the minimal stability margin and maximal instability margin, respectively.

(ii). In Theorem 3.1, condition (3.22) is given to guarantee the stability. Indeed, for any $i \in \mathscr{S}$, there may exist mismatched periods. Those mismatched periods are usually bounded with $d < \infty$. In this case, a larger mode sojourn-time is more appropriate. Based on (3.22), for fixed λ_1, μ and ς, a larger instability margin λ_2 or a larger upper bound on detection delay d can be compensated by a smaller $\bar{\pi}$. By considering $\bar{\pi} = \max_{i\in\mathscr{S}}\{|\pi_{ii}|\}$, one can obtain a smaller $\bar{\pi}$ by decreasing $|\pi_{ii}|$. Then the sojourn-time of $r(t)$ in mode i, $E\{t_{l+1} - t_l | i_l = i, i_{l+1} = j\} = \frac{1}{|\pi_{ii}|}$. Furthermore, one can claim that the average value of the sojourn-time of $r(t)$ is less than or equal to

$\frac{1}{\bar{\pi}}$, and, the smaller $\bar{\pi}$ is, the larger the sojourn-time is. Thus, the stability of the hybrid stochastic retarded systems under asynchronous switching can be guaranteed by a sufficient small detection delay and a sufficient small mode transition rate $\bar{\pi}$. This result has a similar spirit as for asynchronous deterministic switched systems based on average dwell time approach where the closed-loop stability can be guaranteed by a sufficient large average dwell time.

The following two corollaries can be obtained directly from Theorem 3.1 and its proof. Their proofs are omitted.

Corollary 3.1 *System (3.4) under a strictly synchronous controller $\nu(t)$ with $u \equiv 0$ is GASiP if $\mu < \frac{\lambda_1+\tilde{\pi}}{\bar{\pi}}$, and the conditions (3.17)–(3.21) hold.*

Remark 3.4 The similar conclusion can be seen in Corollary 12 in [1], which considers the GAS a.s. of a class of Markovian switching nonlinear systems. Corollary 3.1 provides a sufficient criterion in stochastic case with retarded delays.

Corollary 3.2 *Under the assumptions in Theorem 3.1, system (3.4) with $u \equiv 0$ is also α_1-GASiM. Specially, if $\alpha_1 \in \mathscr{V}\mathscr{K}_\infty$, system (3.4) with $u \equiv 0$ is GASiM. Furthermore, if $\alpha_1(s) = c_1 s^p$, $\alpha_2(s) = c_2 s^p$, where c_1 and c_2 are positive numbers, system (3.4) with $u \equiv 0$ is pth moment exponentially stable.*

Based on the conclusions in Theorem 3.1, we will provide the sufficient conditions of SISS and pth moment ISS for system (3.4).

Theorem 3.2 *System (3.4) is SISS, if (3.17), (3.21) and (3.22) hold and there exist functions $\alpha_1 \in \mathscr{K}_\infty$, $\alpha_2 \in \mathscr{C}\mathscr{K}_\infty$, $\chi \in \mathscr{K}$, scalers $\mu \geq 1$, $q > 1$, $\lambda_1 > 0$, λ_2, $0 < \varsigma < 1$ and $V(x(t), t, \bar{r}(t)) \in \mathscr{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathscr{S} \times \mathscr{S}; \mathbb{R}_+)$, such that for any $l \in \mathbb{N}_+$,*

$$|\varphi(0)| \geq \chi(\|u\|_{[0,\infty)}) \Rightarrow E\{\mathfrak{L}V(\varphi(\theta), t, \bar{r}(t))\}$$
$$\leq \begin{cases} -\lambda_1 E\{V(\varphi(0), t, \bar{r}(t))\}, & t \in [\bar{t}_{2l-2}, \bar{t}_{2l-1}) \\ \lambda_2 E\{V(\varphi(0), t, \bar{r}(t))\}, & t \in [\bar{t}_{2l-1}, \bar{t}_{2l}) \end{cases}$$

provided those $\varphi \in L^p_{\mathscr{F}_t}([-\tau, 0]; \mathbb{R}^n)$ satisfying that (3.19) and (3.20).

Proof Let the time sequences $\{\underline{t}_i\}_{i\geq 1}$ and $\{\tilde{t}_i\}_{i\geq 1}$ denote the time that the trajectory enters and leaves the set $\mathfrak{B} = \{\varphi \in L^p_{\mathscr{F}_t}([-\tau, 0]; \mathbb{R}^n) : |\varphi(0)| < \chi(\|u\|_{[t_0,\infty)})\}$, respectively. In the following, we will complete the proof by considering the following two cases: $\xi \in \mathfrak{B}^C$ and $\xi \in \mathfrak{B} \setminus \{0\}$, respectively.

Case I. $\xi \in \mathfrak{B}^C$.

In this case, for any $t \in [0, \underline{t}_1)$, $|x(t)| \geq \chi(\|u\|_{[0,\infty)})$. According to Theorem 3.1, for any $\varepsilon' > 0$, there exists a $\mathscr{K}\mathscr{L}$ function β such that

$$P\{|x(t)| < \beta(E\{\|\xi\|\}, t)\} \geq 1 - \varepsilon', \ \forall t \in [0, \underline{t}_1). \tag{3.37}$$

Now consider the interval $t \in [\underline{t}_1, \infty)$. Define $\tilde{t}_1 = \inf\{t > \underline{t}_1 : |x(t)| \geq \chi(\|u\|_{[t_0,\infty)})\}$, and $\inf \emptyset = \infty$. Clearly, for any $t \in [\underline{t}_1, \tilde{t}_1)$, we have

$$P\{|x(t)| < \chi(\|u\|_{[0,\infty)})\} = 1 \geq 1 - \varepsilon'', \ \forall \varepsilon'' > 0. \tag{3.38}$$

Define $\underline{t}_2 = \min\{t \geq \tilde{t}_1 : |x(t)| < \chi(\|u\|_{[t_0,\infty)})\}$. According to Theorem 3.1, we also have

$$P\{|x(t)| < \beta(x(\tilde{t}_1), t - \tilde{t}_1)\} \geq 1 - \varepsilon', \ \forall t \in [\tilde{t}_1, \underline{t}_2).$$

Similarly, for any $i \geq 2$, we define

$$\begin{cases} \underline{t}_i = \min\{t \geq \tilde{t}_{i-1} : |x(t)| < \chi(\|u\|_{[t_0,\infty)})\}, \\ \tilde{t}_i = \inf\{t > \underline{t}_i : |x(t)| \geq \chi(\|u\|_{[t_0,\infty)})\}. \end{cases}$$

By repeating the above induction, for any $i \geq 1$, when $t \in [\underline{t}_i, \tilde{t}_i)$, we can obtain

$$P\{|x(t)| < \chi(\|u\|_{[t_0,\infty)})\} = 1 \geq 1 - \varepsilon'',$$

and when $t \in [\tilde{t}_i, \underline{t}_{i+1})$,

$$P\{|x(t)| < \beta(E\{|x(\tilde{t}_i)|\}, t - \tilde{t}_i)\} \geq 1 - \varepsilon'.$$

From the proof of Theorem 3.1, the $\mathscr{KL}$ function $\beta(r, s)$ satisfies

$$\beta(r, s) \leq \alpha_1^{-1}(\bar{M} e^{-\lambda_3 s} \alpha_2(r)),$$

for some $\bar{M} \geq 0$, where $\lambda_3 \in (0, (1 - \varsigma)\bar{\lambda}_1)$. Since $\alpha_1 \in \mathscr{K}_\infty$, further, we can get

$$\beta(r, s) \leq \alpha_1^{-1}(\bar{M} \alpha_2(r)).$$

Thus, for any $i \geq 1$, when $t \in [\underline{t}_i, \tilde{t}_i)$,

$$P\{|x(t)| < \chi(\|u\|_{[t_0,\infty)})\} = 1 \geq 1 - \varepsilon'', \tag{3.39}$$

and when $t \in [\tilde{t}_i, \underline{t}_{i+1})$,

$$\begin{aligned} &P\{|x(t)| < \alpha_1^{-1}(\bar{M} \alpha_2(E\{|x(\tilde{t}_i)|\}))\} \\ &\geq P\{|x| < \beta(E\{|x(\tilde{t}_i)|\}, t - \bar{t}_i)\} \geq 1 - \varepsilon'. \end{aligned} \tag{3.40}$$

Considering the continuity of $x(t)$, we have

$$E\{|x(\tilde{t}_i)|\} < \chi(\|u\|_{[t_0,\infty)}), \ a.s.. \tag{3.41}$$

Substituting (3.41) into (3.39) and (3.40), we obtain

$$P\{|x(t)| < \gamma(\|u\|_{[0,\infty)})\} \geq 1 - \varepsilon''', \ \forall t \geq \underline{t}_1, \tag{3.42}$$

where $\varepsilon''' = \max\{\varepsilon', \varepsilon''\}$, $\gamma(s) = \max\{\chi(s), \alpha_1^{-1}(\bar{S}\alpha_2(s))\}$.

It's easy to verify that $\gamma \in \mathscr{K}$. Then, combining (3.37) and (3.42), we have

$$P\{|x(t)| < \beta(E\{\|\xi\|\}, t) + \gamma(\|u\|_{[0,\infty)})\} \geq 1 - \varepsilon, \tag{3.43}$$

for any $\xi \in \mathfrak{B}^C$, $t \geq 0$, where $\varepsilon = \max\{\varepsilon', \varepsilon'''\}$.

Case II. $\xi \in \mathfrak{B} \setminus \{0\}$.

In this case, $\underline{t}_1 = 0$ a.s. When $t > 0$, we have $P\{t \in (\underline{t}_1, \infty)\} = P\{t \in (t_0, \infty)\} = 1$. Following the proof of Case I, the inequality (3.42) still holds.

$$P\{|x(t)| < \beta(E\{\|\xi\|\}, t) + \gamma(\|u\|_{[0,\infty)})\} \geq P\{|x(t)| < \gamma(\|u\|_{[0,\infty)})\} \geq 1 - \varepsilon''', \tag{3.44}$$

for any $t \in (0, \infty)$.

When $t = 0$, by the definition of the set $\mathfrak{B}$ and the definition of γ, we can obtain

$$P\{|x(0)| < \beta(E\{\|\xi\|\}, 0) + \gamma(\|u\|_{[0,\infty)})\} \geq P\{|x(0)| < \chi(\|u\|_{[0,\infty)})\} = 1,$$

which implies, for any $\varepsilon_1 > 0$,

$$P\{|x(0)| < \beta(E\{\|\xi\|\}, 0) + \gamma(\|u\|_{[0,\infty)})\} \geq 1 - \varepsilon_1, \tag{3.45}$$

Combining (3.44) and (3.45), we have

$$P\{|x(t)| < \beta(E\{\|\xi\|\}, t) + \gamma(\|u\|_{[0,\infty)})\} \geq 1 - \varepsilon, \tag{3.46}$$

for all $t \geq 0$, $\xi \in \mathfrak{B} \setminus \{0\}$, where $\varepsilon = \max\{\varepsilon''', \varepsilon_1\}$.

Combining the proof of Case I and Case II, for any $\varepsilon > 0$, $t \geq 0$ and $\xi \in \mathscr{C}^b_{\mathscr{F}_0}([-\tau, 0]; \mathbb{R}^n)$, we have

$$P\{|x(t)| < \beta(E\{\|\xi\|\}, t) + \gamma(\|u\|_{[0,\infty)})\} \geq 1 - \varepsilon.$$

By causality we get

$$P\{|x(t)| < \beta(E\{\|\xi\|\}, t) + \gamma(\|u\|_{[0,t)})\} \geq 1 - \varepsilon.$$

Thus, we complete the proof.

Remark 3.5 Since the existence of asynchronous period, if $x(t^*) \in \mathfrak{B}$ for some $t^* \geq 0$, we cannot guarantee that $|x(t)| < \chi(\|u\|_{[0,\infty)})$ a.s., for any $t > t^*$. But, from (3.42), it will be upper bounded by $\|u\|_{[0,\infty)}$ in probability.

Similar to Corollary 3.2, we have the following results.

Corollary 3.3 *Under the hypotheses of Theorem 3.2, system (3.4) is also* α_1*-ISSiM. Specially, if* $\alpha_1(s) = c_1 s^p$, $\alpha_2(s) = c_2 s^p$, *where* c_1 *and* c_2 *are positive numbers, system (3.4) is* p*th moment ISS.*

3.3.2 Stability Analysis Under Extended Asynchronous Switching

This section presents the stability criteria for the SSNLRS under extended asynchronous switching controller. By using Razumikhin-type theorem and average dwell time approach, we give the sufficient conditions for internal stability, i.e., globally asymptotically stability in probability and pth moment exponentially stability. Using the internal stability criteria, then the external stability criteria are developed, including SISS and pth moment ISS. Before continuing, some necessary lemmas are stated as follows.

Lemma 3.3 *For any given* $V(x(t), t, r(t), r'(t)) \in \mathscr{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathscr{S} \times \mathscr{S}; \mathbb{R}_+)$*, associated with system (3.4), the diffusion operator* $\mathfrak{L}V$*, from* $\mathscr{C}([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+ \times \mathscr{S} \times \mathscr{S}$ *to* $\mathbb{R}$*, can be described as follows.*

Case I. *When* $r = r' = i$*, we have*

$$\mathfrak{L}V(x_t, t, i, i) = V_t(x, t, i, i) + V_x(x, t, i, i)\bar{f}_{ii}(t, x_t, u) + \sum_{k=1}^{N} \pi_{ik}^1 V(x, t, i, k) + \frac{1}{2} tr[\bar{g}_{ii}^T(t, x_t, u) V_{xx}(x, t, i, i)\bar{g}_{ii}(t, x_t, u)]. \tag{3.47}$$

Case II. *When* $r' = j$, $r = i$ *and* $j \neq i$*, we also have*

$$\begin{aligned}\mathfrak{L}V(x_t, t, i, j) = {} & V_t(x, t, i, j) + V_x(x, t, i, j)\bar{f}_{ij}(t, x_t, u) \\ & + \pi_{ji}^0 V(x, t, i, i) - \pi_{ji}^0 V(x, t, i, j) \\ & + \frac{1}{2} tr[\bar{g}_{ij}^T(t, x_t, u) V_{xx}(x, t, i, j)\bar{g}_{ij}(t, x_t, u)].\end{aligned} \tag{3.48}$$

where $i, j \in \mathscr{S}$*.*

Proof The proof can be got directly from [10, 11].

Lemma 3.4 *[1] Let* $r(t)$ *denote a continuous-time Markov process with transition rate matrix* $[\pi_{ij}]_{N\times N} \in \mathbb{R}^{N\times N}$*, then*

$$P\{N_r(t, 0) = k\} \leq e^{-\tilde{\pi}t} \frac{(\bar{\pi}t)^k}{k!},$$

for any $k \geq 0$*, where* $\bar{\pi} \triangleq \max_{i\in\mathscr{S}}\{|\pi_{ii}|\}$*,* $\tilde{\pi} \triangleq \max_{i,j\in\mathscr{S}}\{\pi_{ij}\}$*, and* $N_r(t,0)$ *denotes the number of switches of* $r(t)$ *on the time-interval* $[0,t]$.

Lemma 3.5 *For any* $i \geq 0$*, we have*

$$P\{N(t_{i+1},t_i)=k\} \leq \begin{cases} e^{-\tilde{\pi}^1(t_{i+1}-t_i)}\frac{(\bar{\pi}^1(t_{i+1}-t_i))^{\frac{k-1}{2}}}{\frac{k-1}{2}!}, \\ \qquad k\, is\, an\, odd\, number \\ e^{-\tilde{\pi}^1(t_{i+1}-t_i)}\frac{(\bar{\pi}^1(t_{i+1}-t_i))^{\frac{k}{2}}}{\frac{k}{2}!}, \\ \qquad k\, is\, an\, even\, number \end{cases}$$

for any $k \in N_+ \cup \{0\}$.

Proof Let $N_1(t_{i+1},t_i)$ denote the numbers of switches from false alarm on time interval $[t_i,t_{i+1})$. In the next, we will complete the proof by considering the following two cases: $N(t_{i+1},t_i)=2k+1$ and $N(t_{i+1},t_i)=2k$, where $k\in N_+\cup\{0\}$. From the Assumption 3.1, one can obtain $N_1(t_{i+1},t_i)=\frac{2k+1-1}{2}=k$ in the first case, while $N_1(t_{i+1},t_i)=\frac{2k}{2}=k$ in the second case. Then, similar to Lemma 3.4, it follows

$$\begin{aligned} P\{N(t_{i+1},t_i)=2k+1\} &\leq P\{N_1(t_{i+1},t_i)=\frac{2k+1-1}{2}\} \\ &\leq e^{-\tilde{\pi}^1(t_{i+1}-t_i)}\frac{(\bar{\pi}^1(t_{i+1}-t_i))^k}{k!}, \end{aligned}$$

and

$$\begin{aligned} P\{N(t_{i+1},t_i)=2k\} &\leq P\{N_1(t_{i+1},t_i)=\frac{2k}{2}\} \\ &\leq e^{-\tilde{\pi}^1(t_{i+1}-t_i)}\frac{(\bar{\pi}^1(t_{i+1}-t_i))^k}{k!}. \end{aligned}$$

Thus we complete the proof.

Lemma 3.6 *For every* $i \geq 0$*, the moment generating function* $E\{e^{sN(t_{i+1},t_i)}\}$ *of* $N(t_{i+1},t_i)$ *satisfies*

$$E\{e^{sN(t_{i+1},t_i)}\} \leq (1+e^s)e^{(e^{2s}\bar{\pi}^1-\tilde{\pi}^1)(t_{i+1}-t_i)},$$

for any $s \geq 0$.

Proof Based on Lemma 3.5, we have

$$\begin{aligned} E\{e^{sN(t_{i+1},t_i)}\} = &\sum_{k=1,3,5,\ldots} e^{sk}P\{N(t_{i+1},t_i)=k\} \\ &+\sum_{k=0,2,4,\ldots} e^{sk}P\{N(t_{i+1},t_i)=k\} \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{k=1,3,5,\ldots} e^{sk} e^{-\tilde{\pi}^1(t_{i+1}-t_i)} \frac{(\bar{\pi}^1(t_{i+1}-t_i))^{\frac{k-1}{2}}}{\frac{k-1}{2}!} \\
&\quad + \sum_{k=0,2,4,\ldots} e^{sk} e^{-\tilde{\pi}^1(t_{i+1}-t_i)} \frac{(\bar{\pi}^1(t_{i+1}-t_i))^{\frac{k}{2}}}{\frac{k}{2}!} \\
&= \sum_{k=0,1,2,\ldots} e^{s(2k+1)} e^{-\tilde{\pi}^1(t_{i+1}-t_i)} \frac{(\bar{\pi}^1(t_{i+1}-t_i))^{k}}{k!} \\
&\quad + \sum_{k=0,1,2,\ldots} e^{2sk} e^{-\tilde{\pi}^1(t_{i+1}-t_i)} \frac{(\bar{\pi}^1(t_{i+1}-t_i))^{k}}{k!} \\
&= (1+e^s)e^{(e^{2s}\bar{\pi}^1-\tilde{\pi}^1)(t_{i+1}-t_i)}.
\end{aligned}
$$

Thus, we complete the proof.

Remark 3.6 Following the proof of Lemma 3.6, it follows

$$
E\{N(t_{i+1}, t_i)\} \le (1+4\bar{\pi}^1\varsigma)e^{\bar{\pi}^1\varsigma},
$$

where $\varsigma = \sup_{l\in\mathbb{N}_+}\{t_l - t_{l-1}\}$.

For the sake of simplifying expression, denote $\bar{\pi}^0 \triangleq \max_{i\in\mathscr{S}}\{|\pi_{ii}^0|\}$, $\tilde{\pi}^0 \triangleq \max_{i,j\in\mathscr{S}}\{\pi_{ij}^0\}$, $\bar{\pi}^1 \triangleq \max_{i\in\mathscr{S}}\{|\pi_{ii}^1|\}$, $\tilde{\pi}^1 \triangleq \max_{i,j\in\mathscr{S}}\{\pi_{ij}^1\}$, $\underline{\pi}^0 \triangleq \min_{i\in\mathscr{S}}\{|\pi_{ii}^0|\}$, $\underline{\pi}^1 \triangleq \min_{i\in\mathscr{S}}\{|\pi_{ii}^1|\}$.

Theorem 3.3 *Let $\varsigma = \sup_{l\in\mathbb{N}_+}\{t_l - t_{l-1}\} < \infty$. If there exist functions $\alpha_1 \in \mathscr{K}_\infty$, $\alpha_2 \in \mathscr{C}\mathscr{K}_\infty$, $\mu \ge 1$, $q > 1$, $\lambda_1 > 0$, $\lambda_2 \ge 0$, and $V(x(t), t, r(t), r'(t)) \in \mathscr{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathscr{S} \times \mathscr{S}; \mathbb{R}_+)$, such that*

(i). for all $t \ge t_0 - \tau$,

$$
\alpha_1(|x(t)|) \le V(x(t), t, r(t), r'(t)) \le \alpha_2(|x(t)|). \tag{3.49}
$$

(ii). there exists $\bar{\lambda}_1 \in (0, \lambda_1)$ such that

$$
\begin{aligned}
&E\{\mathfrak{L}V(\varphi(\theta), t, r(t), r'(t))\} \\
&\le \begin{cases} -\lambda_1 E\{V(\varphi(0), t, r(t), r'(t))\}, \\ \qquad \text{if } t \in T_s(t_l, t_{l+1}),\ l \in \mathbb{N}_+ \cup \{0\} \\ \lambda_2 E\{V(\varphi(0), t, r(t), r'(t))\}, \\ \qquad \text{if } t \in T_a(t_l, t_{l+1}),\ l \in \mathbb{N}_+ \end{cases}
\end{aligned} \tag{3.50}
$$

provided those $\varphi \in L^p_{\mathscr{F}_t}([-\tau, 0]; \mathbb{R}^n)$ satisfying that

$$
\min_{i,j\in\mathscr{S}} E\{V(\varphi(\theta), t+\theta, i, j)\} \le qE\{V(\varphi(0), t, r(t), r'(t))\}, \tag{3.51}
$$

where

$$e^{\bar{\lambda}_1 \tau} \le q. \tag{3.52}$$

(iii). for any $i, j \ge 1$, the candidate function $V(x(t), t, r(t), r'(t))$ satisfies

$$\begin{cases} E\{V(x(t'_{ij}), t'_{ij}, r(t_i), r'(t'_{ij}))\} \\ \qquad \le \mu E\{V(x(t'_{ij}), t'_{ij}, r(t_i), r'(t'_{i(j-1)}))\}, \\ E\{V(x(t_i), t_i, r(t_i), r'(t_i))\} \\ \qquad \le \mu E\{V(x(t_i), t_i, r(t_{i-1}), r'(t'_{(i-1)N(t_i, t_{i-1})}))\}. \end{cases} \tag{3.53}$$

(iv). it exists $\bar{\lambda}_2 \in (\lambda_2, \infty)$ such that

$$\bar{\lambda}_1 + \bar{\lambda}_2 - \underline{\pi}^0 < 0, \tag{3.54}$$

for any $i \ge 1$ and $j = 1, 2, \ldots, N(t_{i+1}, t_i)$, with $t'_{i0} = t_i$, $t'_{0N(t_1,t_0)} = t'_{00} = t_0$, further, the average dwell time τ^* satisfies $\tau^* > \frac{\ln(\mu M)}{\bar{\lambda}_1}$, where

$$M = (1+\mu)[\frac{-\bar{\pi}^0}{\bar{\lambda}_1 + \bar{\lambda}_2 - \underline{\pi}^0} + (\frac{-\bar{\pi}^0}{\bar{\lambda}_1 + \bar{\lambda}_2 - \underline{\pi}^0})^{\frac{1}{2}}] \times e^{[(\mu^2 - \frac{\bar{\pi}^0}{\bar{\lambda}_1 + \bar{\lambda}_2 - \underline{\pi}^0})(N-1) - 2]\tilde{\pi}^1 \varsigma}, \tag{3.55}$$

then system (3.4) with $u \equiv 0$ is GASiP.

Proof According to (3.12) in Lemma 3.2, it has

$$\mathrm{D}^+ E\{V(x, t, r, r')\} = E\{\mathfrak{L}V(x_t, t, r, r')\}, \tag{3.56}$$

for all $t \in T_s(t_l, t_{l+1}) \cup T_a(t_l, t_{l+1}),\ l \in \mathbb{N}_+$.

On the one hand, from (3.49), using Jensen's inequality,

$$E\{V(x, t, i_0, i_0)\} = E\{V(x, t, r, r')\} \le E\{\alpha_2(|x|)\} \le \alpha_2(E\{\|\xi\|\})$$

holds for any $t \in [t_0 - \tau, t_0]$.

In the following, we prove that when $t \in [t_0, t_1)$,

$$E\{V(x, t, i_0, i_0)\} \le \alpha_2(E\{\|\xi\|\}) e^{-\bar{\lambda}_1 (t - t_0)}. \tag{3.57}$$

Suppose (3.57) is not true, i.e., there exists some $t \in (t_0, t_1)$ such that

$$E\{V(x(t), t, i_0, i_0)\} > \alpha_2(E\{\|\xi\|\}) e^{-\bar{\lambda}_1 (t - t_0)}.$$

Let $t^* = \inf\{t \in (t_0, t_1) : E\{V(x(t), t, i_0, i_0)\} > \alpha_2(E\{\|\xi\|\}) e^{-\bar{\lambda}_1 (t-t_0)}\}$. By the continuity of $V(x(t), t, i_0, i_0)$ and $x(t)$ on $[t_0, t_1)$, then we have $t^* \in [t_0, t_1)$ and

$E\{V(x(t^*), t^*, i_0, i_0)\} = \alpha_2(E\{\|\xi\|\})e^{-\bar{\lambda}_1(t^*-t_0)}$. Further, there exists a sequence $\{\tilde{t}_n\}$ ($\tilde{t}_n \in (t^*, t_1)$, for any $n \in \mathbb{N}_+$) with $\lim_{n\to\infty} \tilde{t}_n = t^*$, such that

$$E\{V(x(\tilde{t}_n), \tilde{t}_n), i_0, i_0\} > \alpha_2(E\{\|\xi\|\})e^{-\bar{\lambda}_1(\tilde{t}_n-t_0)}. \tag{3.58}$$

From the definition of t^*, we have

$$\begin{aligned} E\{V(x(t^*+\theta), t^*+\theta, i_0, i_0)\} &\le e^{-\bar{\lambda}_1\theta} E\{V(x(t^*), t^*, i_0, i_0)\} \\ &\le qE\{V(x(t^*), t^*, i_0, i_0)\}, \end{aligned}$$

and further,

$$\min_{i,j\in\mathscr{S}} E\{V(x(t^*+\theta), t^*+\theta, i, j)\} \le qE\{V(x(t^*), t^*, i_0, i_0)\},$$

for any $\theta \in [-\tau, 0]$.

Then, based on (3.50) and (3.51), the following equation holds

$$\mathrm{D}^+ E\{V(x(t^*), t^*, i_0, i_0)\} \le -\lambda_1 E\{V(x(t^*), t^*, i_0, i_0)\}. \tag{3.59}$$

Without loss of generality, we have

$$\begin{aligned} \mathrm{D}^+ E\{V(x(t^*), t^*, i_0, i_0)\} &\le -\lambda_1 E\{V(x(t^*), t^*, i_0, i_0) \\ &< -\bar{\lambda}_1 E\{V(x(t^*), t^*, i_0, i_0)\}. \end{aligned}$$

For $h > 0$ which is sufficient small, when $t \in [t^*, t^*+h]$, it follows

$$\mathrm{D}^+ E\{V(x(t), t, i_0, i_0)\} \le -\bar{\lambda}_1 E\{V(x(t), t, i_0, i_0)\},$$

which means

$$E\{V(x(t^*+h), t^*+h, i_0, i_0)\} \le E\{V(x(t^*), t^*, i_0, i_0)\}e^{-\bar{\lambda}_1 h},$$

and it is a contradiction of (3.58), thus (3.57) holds.

Combining the continuity of function $V(x(t), t, i_0, i_0)$ and (3.53), we have

$$\begin{aligned} E\{V(x(t_1), t_1, r(t_1), r'(t_1))\} &\le \mu E\{V(x(t_1), t_1, i_0, i_0)\} \\ &\le \mu\alpha_2(E\{\|\xi\|\})e^{-\bar{\lambda}_1(t_1-t_0)}. \end{aligned} \tag{3.60}$$

On the other hand, let $W(t, \bar{r}(t)) = W(t, r(t), r'(t)) = e^{\bar{\lambda}_1 t} V(x(t), t, r(t), r'(t))$.

Then, for any $l \in N_+$ and $\theta \in [-\tau, 0]$, we have

$$\mathrm{D}^+ E\{W(t, \bar{r}(t))\} \leq \begin{cases} -(\lambda_1 - \bar{\lambda}_1) E\{W(t, \bar{r}(t))\}, \\ \qquad t \in T_s(t_l, t_{l+1}) \\ (\bar{\lambda}_1 + \lambda_2) E\{W(t, \bar{r}(t))\}, \\ \qquad t \in T_a(t_l, t_{l+1}) \end{cases}$$

whenever (3.51) holds.

For any $[s_1, s_2) \subset T_a(t_l, t_{l+1})$, we claim that when $t \in [s_1, s_2)$,

$$E\{W(t, \bar{r}(t))\} \leq e^{(\bar{\lambda}_1 + \bar{\lambda}_2)(t - s_1)} E\{W(s_1, \bar{r}(s_1))\}. \tag{3.61}$$

Suppose (3.61) is not true, i.e., there exists some $t \in [s_1, s_2)$ such that

$$E\{W(t, \bar{r}(t))\} > e^{(\bar{\lambda}_1 + \bar{\lambda}_2)(t - s_1)} E\{W(s_1, \bar{r}(s_1))\}.$$

Similarly, set

$$t^* = \inf\{t \in [s_1, s_2) : E\{W(t, \bar{r}(t))\} > E\{W(s_1, \bar{r}(s_1))\} \times \ e^{(\bar{\lambda}_1 + \bar{\lambda}_2)(t - s_1)}\},$$

Then

$$E\{W(t^*, \bar{r}(t^*))\} = E\{W(s_1, \bar{r}(s_1))\} e^{(\bar{\lambda}_1 + \bar{\lambda}_2)(t^* - s_1)}.$$

Moreover, there is a sequence $\{\tilde{t}_n\}_{n \in \mathbb{N}_+} \in (t^*, s_2)$ with $\lim_{n \to \infty} \tilde{t}_n = t^*$ such that

$$\begin{aligned} E\{W(\tilde{t}_n, \bar{r}(\tilde{t}_n))\} &> E\{W(s_1, \bar{r}(s_1))\} e^{(\bar{\lambda}_1 + \bar{\lambda}_2)(\tilde{t}_n - s_1)} \\ &= E\{W(t^*, \bar{r}(t^*))\} e^{(\bar{\lambda}_1 + \bar{\lambda}_2)(\tilde{t}_n - t^*)}. \end{aligned} \tag{3.62}$$

We further define $U(t) = e^{-(\bar{\lambda}_1 + \bar{\lambda}_2)t} W(t, \bar{r}(t))$, then

$$\mathrm{D}^+ E\{U(t)\} = -\bar{\lambda}_2 e^{-\bar{\lambda}_2 t} E\{V(x(t), t, \bar{r}(t))\} + e^{-\bar{\lambda}_2 t} \mathrm{D}^+ E\{V(x(t), t, \bar{r}(t))\}.$$

From the definition of t^*, for any $\theta \in [-\tau, 0]$, it follows

$$\begin{aligned} E\{W(t^*, \bar{r}(t^*))\} e^{(\bar{\lambda}_1 + \bar{\lambda}_2)\theta} &= E\{W(s_1, \bar{r}(s_1))\} e^{(\bar{\lambda}_1 + \bar{\lambda}_2)(t^* + \theta - s_1)} \\ &\geq E\{W(t^* + \theta, \bar{r}(t^* + \theta))\}, \end{aligned}$$

which means

$$\begin{aligned} E\{V(x(t^* + \theta), t^* + \theta, \bar{r}(t^* + \theta))\} &\leq E\{V(x(t^*), t^*, \bar{r}(t^*))\} e^{\bar{\lambda}_2 \theta} \\ &\leq E\{V(x(t^*), t^*, \bar{r}(t^*))\}, \end{aligned}$$

and further

$$\min_{i,j\in\mathscr{S}} E\{V(x(t^*+\theta), t^*+\theta, i, j)\} \leq qE\{V(x(t^*), t^*, \bar{r}(t^*))\}.$$

Then, from (3.50) and (3.51), we have

$$\mathrm{D}^+E\{U(t^*)\} \leq -(\bar{\lambda}_2-\lambda_2)e^{-\bar{\lambda}_2 t^*}E\{V(x(t^*), t^*, \bar{r}(t^*))\}.$$

Without loss of generality, it follows

$$\mathrm{D}^+E\{U(t^*)\} < 0.$$

Moreover, there exists a positive number h which is sufficient small such that

$$\mathrm{D}^+E\{U(t)\} \leq 0,\ t \in [t^*, t^*+h].$$

It then follows

$$E\{W(t^*+h, \bar{r}(t^*+h))\} \leq E\{W(t^*, \bar{r}(t^*))\}e^{(\bar{\lambda}_1+\bar{\lambda}_2)h},$$

which is a contradiction of (3.62). Thus, (3.61) is true.

Furthermore, when $t \in [s_1, s_2) \in T_s(t_l, t_{l+1})$, repeating a similar analysis (similar to the proof of (3.57)), one can obtain

$$E\{W(t, \bar{r}(t))\} \leq E\{W(s_1, \bar{r}(s_1))\}. \tag{3.63}$$

Combining (3.61) and (3.63), if the detection delay is non-zero, it holds

$$E\{W(t, \bar{r}(t))\} \leq \begin{cases} e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t-t_l)}E\{W(t_l, \bar{r}(t_l))\}, & t \in [t_l, t'_{l1}) \\ E\{W(t'_{l1}, \bar{r}(t'_{l1}))\}, & t \in [t'_{l1}, t'_{l2}) \\ e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t-t'_{l2})}E\{W(t'_{l2}, \bar{r}(t'_{l2}))\}, & t \in [t'_{l2}, t'_{l3}) \\ E\{W(t'_{l3}, \bar{r}(t'_{l3}))\}, & t \in [t'_{l3}, t'_{l4}) \\ \cdots & \\ e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t-t'_{l(N(t_{l+1},t_l)-1)})} & \\ \quad \times E\{W(t'_{l(N(t_{l+1},t_l)-1)}, \bar{r}(t'_{l(N(t_{l+1},t_l)-1)}))\}, & \\ \qquad t \in [t'_{l(N(t_{l+1},t_l)-1)}, t'_{lN(t_{l+1},t_l)}) & \\ E\{W(t'_{lN(t_{l+1},t_l)}, \bar{r}(t'_{lN(t_{l+1},t_l)}))\}, & \\ \qquad t \in [t'_{lN(t_{l+1},t_l)}, t_{l+1}) & \end{cases} \tag{3.64}$$

and in this case, $N(t_{l+1}, t_l)$ is an even number. If the detection delay is equal to zero, it also has

$$
E\{W(t,\bar r(t))\}
\leq \begin{cases}
E\{W(t_l,\bar r(t_l))\}, & t\in[t_l,t'_{l1})\\
e^{(\bar\lambda_1+\bar\lambda_2)(t-t'_{l1})}E\{W(t'_{l1},\bar r(t'_{l1}))\}, & t\in[t'_{l1},t'_{l2})\\
E\{W(t'_{l2},\bar r(t'_{l2}))\}, & t\in[t'_{l2},t'_{l3})\\
e^{(\bar\lambda_1+\bar\lambda_2)(t-t'_{l3})}E\{W(t'_{l3},\bar r(t'_{l3}))\}, & t\in[t'_{l3},t'_{l4})\\
\cdots\\
e^{(\bar\lambda_1+\bar\lambda_2)(t-t'_{l(N(t_{l+1},t_l)-1)})}\\
\quad\times E\{W(t'_{l(N(t_{l+1},t_l)-1)},\bar r(t'_{l(N(t_{l+1},t_l)-1)}))\},\\
\qquad t\in[t'_{l(N(t_{l+1},t_l)-1)},t'_{lN(t_{l+1},t_l)})\\
E\{W(t'_{lN(t_{l+1},t_l)},\bar r(t'_{lN(t_{l+1},t_l)}))\},\\
\qquad t\in[t'_{lN(t_{l+1},t_l)},t_{l+1})
\end{cases}
\tag{3.65}
$$

and in this case, $N(t_{l+1},t_l)$ is an odd number.

Then, for any $t\in[t_l,t_{l+1})$, if $[t'_{lN(t,t_l)},t'_{l(N(t,t_l)+1)})\in T_s(t_l,t_{l+1})$, we can obtain

$$
\begin{aligned}
E\{W(t,\bar r(t))\} &\leq E\{W(t'_{lN(t,t_l)},\bar r(t'_{lN(t,t_l)}))\}\\
&\leq E\{\mu W(t'_{lN(t,t_l)},\bar r(t'_{l(N(t,t_l)-1)}))\}\\
&= E\{\mu^{N(t,t_l)-N(t,t_l)+1}\}E\{W(t'_{lN(t,t_l)},\bar r(t'_{l(N(t,t_l)-1)}))\}\\
&\leq E\{\mu^{N(t,t_l)-N(t,t_l)+1}\}E\{e^{(\bar\lambda_1+\bar\lambda_2)(t'_{lN(t,t_l)}-t'_{l(N(t,t_l)-1)})}\\
&\quad\times W(t'_{l(N(t,t_l)-1)},\bar r(t'_{l(N(t,t_l)-1)}))\}\\
&\leq E\{\mu^{N(t,t_l)-N(t,t_l)+2}\}E\{e^{(\bar\lambda_1+\bar\lambda_2)(t'_{lN(t,t_l)}-t'_{l(N(t,t_l)-1)})}\}\\
&\quad\times E\{W(t'_{l(N(t,t_l)-1)},\bar r(t'_{l(N(t,t_l)-2)}))\}\\
&\leq E\{\mu^{N(t,t_l)-N(t,t_l)+2}\}E\{e^{(\bar\lambda_1+\bar\lambda_2)(t'_{lN(t,t_l)}-t'_{l(N(t,t_l)-1)})}\}\\
&\quad\times E\{W(t'_{l(N(t,t_l)-2)},\bar r(t'_{l(N(t,t_l)-2)}))\}\\
&\leq E\{\mu^{N(t,t_l)-N(t,t_l)+3}\}E\{e^{(\bar\lambda_1+\bar\lambda_2)(t'_{lN(t,t_l)}-t'_{l(N(t,t_l)-1)})}\}\\
&\quad\times E\{W(t'_{l(N(t,t_l)-2)},\bar r(t'_{l(N(t,t_l)-3)}))\}\\
&\leq E\{\mu^{N(t,t_l)-N(t,t_l)+3}\}E\{e^{(\bar\lambda_1+\bar\lambda_2)(t'_{lN(t,t_l)}-t'_{l(N(t,t_l)-1)})}\\
&\quad\times e^{(\bar\lambda_1+\bar\lambda_2)(t'_{l(N(t,t_l)-2)}-t'_{l(N(t,t_l)-3)})}\}\\
&\quad\times E\{W(t'_{l(N(t,t_l)-3)},\bar r(t'_{l(N(t,t_l)-3)}))\}\\
&\leq\cdots\\
&\leq E\{\mu^{N(t,t_l)}\}E\{e^{(\bar\lambda_1+\bar\lambda_2)T_a(t-t_l)}\}E\{W(t_l,\bar r(t_l))\}.
\end{aligned}
$$

And similarly, on the one hand, if $[t'_{lN(t,t_l)},t'_{l(N(t,t_l)+1)})\in T_a(t_l,t_{l+1})$, it also follows that

$$
\begin{aligned}
&E\{W(t,\bar r(t))\}\\
&\leq E\{\mu^{N(t,t_l)}\}E\{e^{(\bar\lambda_1+\bar\lambda_2)T_a(t-t_l)}\}E\{W(t_l,\bar r(t_l))\}.
\end{aligned}
$$

Then, without loss of generality, for any $t \in [t_l, t_{l+1})$, it holds

$$\begin{aligned}
&E\{W(t, \bar{r}(t))\} \\
&\leq E\{\mu^{N(t,t_l)}\}E\{e^{(\bar{\lambda}_1+\bar{\lambda}_2)T_a(t-t_l)}\}E\{W(t_l, \bar{r}(t_l))\} \\
&\leq E\{\mu^{N(t_{l+1},t_l)}\}E\{e^{(\bar{\lambda}_1+\bar{\lambda}_2)T_a(t_{l+1}-t_l)}\}E\{W(t_l, \bar{r}(t_l))\}. \qquad (3.66)
\end{aligned}$$

On the other hand, for any $l \geq 0$, it holds (3.67a) and (3.67b).

$$E\{e^{(\bar{\lambda}_1+\bar{\lambda}_2)T_a(t_{l+1}-t_l)}\}$$

$$= \begin{cases} E\{\underbrace{e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t'_{l1}-t'_{l0})}e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t'_{l3}-t'_{l2})}\cdots e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t'_{lN(t_{l+1},t_l)}-t'_{l(N(t_{l+1},t_l)-1)})}}_{\frac{N(t_{l+1},t_l)+1}{2},\ N(t_{l+1},t_l)\ is\ an\ odd\ number}\} & (3.67a) \\ E\{\underbrace{e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t'_{l2}-t'_{l1})}e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t'_{l4}-t'_{l3})}\cdots e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t'_{lN(t_{l+1},t_l)}-t'_{l(N(t_{l+1},t_l)-1)})}}_{\frac{N(t_{l+1},t_l)}{2},\ N(t_{l+1},t_l)\ is\ an\ even\ number}\} & (3.67b) \end{cases}$$

Since

$$\begin{aligned}
E\{e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t'_{lj}-t'_{lj-1})}\} &\leq \int_0^\infty e^{(\bar{\lambda}_1+\bar{\lambda}_2)t}\bar{\pi}^0\, e^{-\underline{\pi}^{0t}}\,\mathrm{d}t \\
&= \frac{-\bar{\pi}^0}{\bar{\lambda}_1+\bar{\lambda}_2-\underline{\pi}^0},
\end{aligned}$$

then, based on Lemma 3.6, let $s = \ln\sqrt{\frac{-\bar{\pi}^0}{\bar{\lambda}_1+\bar{\lambda}_2-\underline{\pi}^0}}$, we have

$$\begin{aligned}
&E\{e^{(\bar{\lambda}_1+\bar{\lambda}_2)T_a(t_{l+1}-t_l)}\} \\
&\leq \begin{cases} E\{(\frac{-\bar{\pi}^0}{\bar{\lambda}_1+\bar{\lambda}_2-\underline{\pi}^0})^{\frac{N(t_{l+1},t_l)+1}{2}}\}, \\ \qquad N(t_{l+1},t_l) \text{ is an odd number} \\ E\{(\frac{-\bar{\pi}^0}{\bar{\lambda}_1+\bar{\lambda}_2-\underline{\pi}^0})^{\frac{N(t_{l+1},t_l)}{2}}\}, \\ \qquad N(t_{l+1},t_l) \text{ is an even number} \end{cases} \\
&\leq \begin{cases} (\sqrt{K}_1+K_1)e^{(K_1\bar{\pi}^1-\tilde{\pi}^1)(t_{l+1}-t_l)}, \\ \qquad N(t_{l+1},t_l) \text{ is an odd number} \\ (1+\sqrt{K}_1)e^{(K_1\bar{\pi}^1-\tilde{\pi}^1)(t_{l+1}-t_l)}, \\ \qquad N(t_{l+1},t_l) \text{ is an even number} \end{cases} \qquad (3.68)
\end{aligned}$$

and without loss of generality,

$$E\{e^{(\bar{\lambda}_1+\bar{\lambda}_2)T_a(t_{l+1}-t_l)}\} \leq K_2 e^{(K_1\bar{\pi}^1-\tilde{\pi}^1)(t_{l+1}-t_l)},$$

where $K_1 = \frac{-\bar{\pi}^0}{\bar{\lambda}_1+\bar{\lambda}_2-\underline{\pi}^0}$, $K_2 = K_1 + \sqrt{K}_1 = \max\{\sqrt{K}_1 + K_1, 1 + \sqrt{K}_1\}$.

In addition, if we let $s = \ln(\mu)$, utilizing the Lemma 3.6 again, we can obtain

$$E\{\mu^{N(t_{l+1},t_l)}\} \leq (1+\mu)e^{(\mu^2\bar{\pi}^1-\tilde{\pi}^1)(t_{l+1}-t_l)}.$$

Consequently, for any $t \in [t_l, t_l + 1)$, it has

$$\begin{aligned} E\{W(t,\bar{r}(t))\} &\leq K_3 e^{k_1(t_{l+1}-t_l)} E\{W(t_l,\bar{r}(t_l))\} \\ &\leq K_3 e^{\bar{k}_1(t_{l+1}-t_l)} E\{W(t_l,\bar{r}(t_l))\} \\ &\leq M E\{W(t_l,\bar{r}(t_l))\}, \end{aligned} \tag{3.69}$$

where $K_3 = (1+\mu)K_2$, $k_1 = (\mu^2 + K_1)\bar{\pi}^1 - 2\tilde{\pi}^1$, $k_1 = [(\mu^2 + K_1)(N-1) - 2]$ $\tilde{\pi}^1$, $M = K_3 e^{\bar{k}_1 \varsigma}$.

From (3.53), for any $t \geq t_1$, iterating (3.69) from $l = 1$ to $l = N_r(t, t_1) + 1$, we can get

$$\begin{aligned} E\{W(t,\bar{r}(t))\} &\leq M E\{W(t_{N_r(t,t_1)+1}, \bar{r}(t_{N_r(t,t_1)+1}))\} \\ &\leq \mu M^2 E\{W(t_{N_r(t,t_1)}, \bar{r}(t_{N_r(t,t_1)}))\} \\ &\leq \mu^2 M^3 E\{W(t_{N_r(t,t_1)-1}, \bar{r}(t_{N_r(t,t_1)-1}))\} \\ &\leq \cdots \\ &\leq \mu^{N_r(t,t_1)} M^{N_r(t,t_1)+1} E\{W(t_1,\bar{r}(t_1))\}, \end{aligned}$$

which means for any $t \geq t_1$,

$$\begin{aligned} E\{V(x(t),t,r(t),r'(t))\} \leq \mu^{N_r(t,t_1)} M^{N_r(t,t_1)+1} e^{-\bar{\lambda}_1(t-t_1)} \\ \times E\{V(x(t_1),t_1,r(t_1),r'(t_1))\}. \end{aligned} \tag{3.70}$$

Combining (3.60) and (3.70), we have

$$\begin{aligned} &E\{V(x(t),t,r(t),r'(t))\} \\ &\leq \mu^{N_r(t,t_1)+1} M^{N_r(t,t_1)+1} e^{-\bar{\lambda}_1(t-t_0)} \alpha_2(E\{\|\xi\|\}) \\ &= (\mu M)^{N_r(t,t_0)} e^{-\bar{\lambda}_1(t-t_0)} \alpha_2(E\{\|\xi\|\}) \\ &\leq (\mu M)^{N_0} e^{(-\bar{\lambda}_1+\frac{\ln(\mu M)}{\tau^*})(t-t_0)} \alpha_2(E\{\|\xi\|\}) \\ &\triangleq \tilde{\beta}(E\{\|\xi\|\}, t-t_0), \end{aligned} \tag{3.71}$$

for any $t \geq t_1$.

Clearly, $\tilde{\beta}(\cdot,\cdot) \in \mathscr{K}\mathscr{L}$ if and only if $\tau^* > \frac{\ln(\mu M)}{\bar{\lambda}_1}$. For any $\varepsilon \in (0,1)$, take $\bar{\beta} = \frac{\tilde{\beta}}{\varepsilon} \in \mathscr{K}\mathscr{L}$. Obviously, (3.71) also holds for $t \in [t_0, t_1)$. Then, using Chebyshev's inequality and the above inequality, for all $t \geq t_0$,

$$\begin{aligned} &P\{V(x(t), t, r(t), r'(t)) \geq \bar{\beta}(E\{\|\xi\|\}, t-t_0)\} \\ &\leq \frac{E\{V(x(t), t, r(t), r'(t))\}}{\bar{\beta}(E\{\|\xi\|\}, t-t_0)} < \varepsilon. \end{aligned}$$

Define $\beta(r,s) = \alpha_1^{-1} \circ \bar{\beta}(r,s)$, then

$$P\{|x(t)| < \beta(E\{\|\xi\|\}, t-t_0)\} \geq 1-\varepsilon, \ \forall t \geq t_0,$$

where $\beta(\cdot,\cdot) \in \mathscr{K}\mathscr{L}$.

Thus, we complete the proof.

Remark 3.7 In Theorem 3.3, the assumptions (3.49), (3.51), (3.52) and (3.53) are common conditions in the stability analysis of switched stochastic time-delay systems [15]. The condition (3.50) is also commonly employed in the asynchronous switched deterministic systems [24], while (3.54) is set to restrict the conditions that the system (3.4) needs to be satisfied under the existence of detection delay and false alarm.

Remark 3.8 For the detection of $r(t)$, consider the following two special cases. First, if Π^0 and Π^1 are set to ∞ and zero, respectively, there is no detection delay and no false alarm in the mode detection, the closed-loop system is a synchronous case. In this case, the conditions (3.50) and (3.54) hold almost surely. Second, if Π^1 is set to zero while $\Pi^0 < \infty$, the situation corresponds to only a detection delay, and $\underline{\pi}^0 < \infty$, $\tilde{\pi}^1 = 0$. Hypothesis (3.54) restricts the necessary condition that the closed-loop systems need to be satisfied under this case.

Remark 3.9 According to (3.54), (3.55) and the average dwell time τ^* in Theorem 3.3, one can see that the stability of the extended asynchronous switching systems can be guaranteed by a sufficient small mismatched time interval and a sufficient large average dwell time. Given that the mismatched time interval in the developed extended asynchronous switching framework is usually caused by: the size of detection delay, the frequency of occurrence from false alarms, and the length of the recovery time from a false mode, it is further explained as follows:

(i). For any fixed $\bar{\lambda}_1$, μ, ς and $\tilde{\pi}^1$, a larger instability margin λ_2 (or $\bar{\lambda}_2$) will be compensated by a larger $\underline{\pi}^0$ and/or a larger average dwell time τ^*. Since $\underline{\pi}^0 = \min_{i\in\mathscr{S}}\{|\pi_{ii}^0|\}$, a larger $\underline{\pi}^0$ can be obtained by increasing $|\pi_{ii}^0|$ or decreasing π_{ii}^0. The larger $|\pi_{ii}^0|$ is the smaller of the detection delay for mode i is. Thus, when π_{ii}^0 increases, if $\bar{\pi}^0 = \max_{i,s\in\mathscr{S}}\{\pi_{ij}^0\}$ is non-increase, a lager instability margin can be compensated by a small detection delay; however if $\bar{\pi}^0$ is also increased, a larger average dwell time τ^* will work, and the larger instability margin will be compensated by a smaller detection delay and a larger average dwell time τ^*.

(ii). When $\bar{\lambda}_1$, μ and ς are fixed, and we assume $\underline{\pi}^0$ and $\bar{\pi}^0$ do not change through a fixed constant π_{ij}^0 $(i, j \in \mathscr{S})$, then if the instability margin λ_2 (or $\bar{\lambda}_2$) increases, M will also increase. In this case, the larger M can be compensated by a smaller $\tilde{\pi}^1$ or a larger average dwell time τ^*. Note that, a fixed constant π_{ij}^0 $(i, j \in \mathscr{S})$ means that the time costs of the detection of true modes and the recovery from a false mode do not change on the average. Given that $\tilde{\pi}^1 = \max_{i,j\in\mathscr{S}}\{\pi_{ij}^1\}$, $\tilde{\pi}^1$ can be reduced by decreasing π_{ij}^1. Then the number of false alarms will decrease, and consequently, the mismatched time from false alarms will also decrease, which can well compensate the impact of larger instability margin.

Using the GASiP criterion in Theorem 3.3, one can further obtain the following SISS conditions.

Theorem 3.4 *Let $\varsigma = \sup_{l\in\mathbb{N}_+}\{t_l - t_{l-1}\} < \infty$. If there exist functions $\gamma \in \mathscr{K}$, $\alpha_1 \in \mathscr{V}\mathscr{K}_\infty$, $\alpha_2 \in \mathscr{C}\mathscr{K}_\infty$, $\mu \geq 1$, $q > 1$, $\lambda_1 > 0$, $\lambda_2 \geq 0$, $\bar{\lambda}_1 \in (0, \lambda_1)$, $\bar{\lambda}_2 \in (\lambda_2, \infty)$, and $V(x(t), t, r(t), r'(t)) \in \mathscr{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathscr{S} \times \mathscr{S}; \mathbb{R}_+)$, such that hypotheses (i), (iii), (iv) in Theorem 3.3 hold, and*

$$
\begin{aligned}
|\varphi(0)| \geq \gamma(\|u\|_{[t_0,\infty)}) \Rightarrow E\{\mathfrak{L}V(\varphi(\theta), t, r(t), r'(t))\} \\
\leq \begin{cases} -\lambda_1 E\{V(\varphi(0), t, r(t), r'(t))\}, \\ \qquad t \in T_s(t_l, t_{l+1}),\ l \in \mathbb{N}_+ \cup \{0\} \\ \lambda_2 E\{V(\varphi(0), t, r(t), r'(t))\}, \\ \qquad t \in T_a(t_l, t_{l+1}),\ l \in \mathbb{N}_+ \end{cases}
\end{aligned}
\tag{3.72}
$$

provided those $\varphi \in L^p_{\mathscr{F}_t}([-\tau, 0]; \mathbb{R}^n)$ satisfying that

$$
\min_{i,j\in\mathscr{S}} E\{V(\varphi(\theta), t+\theta, i, j)\} \leq qE\{V(\varphi(0), t, r(t), r'(t))\}, \tag{3.73}
$$

where

$$
e^{\bar{\lambda}_1 \tau} \leq q. \tag{3.74}
$$

Then, system (3.4) is SISS.

Proof The proof is similar to Theorem 3.2 and is thus omitted.

Remark 3.10 Despite the similarities of Theorems 3.2 and 3.4 in this section, the following essential differences are observed.

(i). Due to the existence of mismatched time interval which caused by detection delays and false alarms, after the state trajectory enters the set $\mathfrak{B}$, there still exists a chance to leave it. This complicates the system and is different from the normal asynchronous case in Sect. 3.3.1.

(ii). The system in this section is deterministic switched system while the system in Sect. 3.3.1 is Markovian switching.

(iii). Section 3.3.1 considers only the detection delay while this work consider both the non-zero detection delay and the false alarm. The inclusion of false alarm makes the extended asynchronous switching model in Sect. 3.3.2 more practical.

Corollary 3.4 *Under the assumptions in Theorem 3.3 (Theorem 3.4), if functions α_1, α_2 satisfy $\alpha_1(s) = c_1 s^p$, $\alpha_2(s) = c_2 s^p$, where c_1 and c_2 are positive numbers, then system (3.4) is pth moment exponentially stable with $u \equiv 0$ (pth moment ISS), for all $\tau^* > \frac{\ln(\mu M)}{\bar{\lambda}_1}$.*

3.4 Numerical Simulation

3.4.1 Asynchronous Switching

Hybrid stochastic delay system (HSDS), described by stochastic differential delay equations with Markovian switching, is an important class of hybrid stochastic retarded systems and is frequently used in engineering. In this section, the conclusions established in previous sections are applied to the stability analysis of a class of HSDSs under asynchronous switching.

Consider the following hybrid system which has been discussed in [2] and the reference therein.

$$\begin{cases} \mathrm{d}x(t) = F(t, x(t), x(t - d_1(t, r(t))), \nu(t), r(t))\mathrm{d}t \\ \qquad\quad +G(t, x(t), x(t - d_1(t, r(t))), \nu(t), r(t))\mathrm{d}B(t), \; t \geq 0 \\ \nu(t) = H(t, x(t), u(t), r'(t)), \end{cases} \tag{3.75}$$

where $d_1 : \mathbb{R}_+ \times \mathscr{S} \to [0, \tau]$ is Borel measurable while F, G and H are measurable functions with $F(t, 0, 0, 0, i) \equiv 0$, $G(t, 0, 0, 0, i) \equiv 0$ and $H(t, 0, 0, i) \equiv 0$, for all $t \geq 0$ and $i \in \mathscr{S}$. Let

$$\begin{aligned} &\bar{F}(t, x(t), x(t - d_1(t, r(t))), u(t), \bar{r}(t)) \\ =&F(t, x(t), x(t - d_1(t, r(t))), H(t, x(t), u(t), r'(t)), r(t)) \\ &\bar{G}(t, x(t), x(t - d_1(t, r(t))), u(t), \bar{r}(t)) \\ =&G(t, x(t), x(t - d_1(t, r(t))), H(t, x(t), u(t), r'(t)), r(t)) \end{aligned}$$

and $d_{1r(t)}(t) = d_1(t, r(t))$.

We assume $\bar{F}$ and $\bar{G}$ satisfy the local Lipschitz condition and the linear growth condition. Then, the closed-loop system

$$\begin{aligned} \mathrm{d}x(t) = &\bar{F}_{ij}(t, x(t), x(t - d_{1i}(t)), u(t))\mathrm{d}t \\ &+ \bar{G}_{ij}(t, x(t), x(t - d_{1i}(t)), u(t))\mathrm{d}B(t) \end{aligned} \tag{3.76}$$

has unique solution on $t \geq -\tau$.

In fact, we find that system (3.76) is a special case of (3.4) when $\bar{f}_{ij}(t,\varphi(0),\varphi,u) = \bar{F}_{ij}(t,\varphi(0),\varphi(-d_{1i}(t)),u)$ and $\bar{g}_{ij}(t,\varphi(0),\varphi,u) = \bar{G}_{ij}(t,\varphi(0),\varphi(-d_{1i}(t)),u)$ for $(\varphi,t,i,j) \in \mathscr{C}([-\tau,0];\mathbb{R}^n) \times \mathbb{R}_+ \times \mathscr{S} \times \mathscr{S}$.

In the following, we use Theorem 3.2 to establish a useful stability criterion for system (3.76).

Corollary 3.5 *System (3.76) is SISS if there exist functions $\alpha_1 \in \mathscr{K}_\infty$, $\alpha_2 \in \mathscr{C}\mathscr{K}_\infty$, $\chi \in \mathscr{K}$, scalars $\mu \geq 1$, $q > 1$, $\lambda_k > 0$, $\lambda_{k1} > 0$, $k = 1, 2$, $0 < \varsigma < 1$ and $V(x(t),t,\bar{r}(t)) \in \mathscr{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathscr{S} \times \mathscr{S};\mathbb{R}_+)$, such that (3.17) and (3.21) hold and for any $l \in \mathbb{N}_+$,*

$$\begin{aligned}\mathfrak{L}V(x(t),y_1(t),t,\bar{r}(t)) \leq & -\lambda_1 V(x(t),t,\bar{r}(t)) \\ & + \lambda_{11} \min_{m,n\in\mathscr{S}} \{V(y_1(t),t-d_{1i_l}(t),m,n)\} \\ & + \chi(\|u\|_{[0,\infty)}),\ t \in [\bar{t}_{2l-2},\bar{t}_{2l-1}),\end{aligned} \tag{3.77}$$

and

$$\begin{aligned}\mathfrak{L}V(x(t),y_1(t),t,\bar{r}(t)) \leq & \lambda_2 V(x(t),t,\bar{r}(t)) \\ & + \lambda_{21} \min_{m,n\in\mathscr{S}} \{V(y_1(t),t-d_{1i_l}(t),m,n)\} \\ & + \chi(\|u\|_{[0,\infty)}),\ t \in [\bar{t}_{2l-1},\bar{t}_{2l}),\end{aligned} \tag{3.78}$$

where $y_1(t) = x(t-d_1(t,r(t)))$; and there exists $\lambda_0 > 0$, and $\bar{\lambda}_1 = \lambda_1 - q\lambda_{11} - \lambda_0 > 0$, $\bar{\lambda}_2 = \lambda_2 + q\lambda_{21} + \lambda_0 > 0$, $\hat{\lambda}_1 \in (0,\bar{\lambda}_1)$ and $\hat{\lambda}_2 \in (\bar{\lambda}_2,\infty)$ such that

$$e^{\hat{\lambda}_1\tau} \leq q, \tag{3.79}$$

and

$$\mu^2\bar{\pi}e^{(\hat{\lambda}_1+\hat{\lambda}_2)d} - \tilde{\pi} \leq \varsigma\hat{\lambda}_1. \tag{3.80}$$

Proof From (3.77) and (3.78), there exists $0 < \lambda_0 < \lambda_1$ such that

$$\begin{aligned}|x(t)| \geq \bar{\chi}(\|u\|_{[0,\infty)}) \Rightarrow & \mathfrak{L}V(x(t),y_1(t),t,\bar{r}(t)) \\ & \leq \lambda_{11} \min_{m,n\in\mathscr{S}} V(y_1(t),t-d_{1i_l}(t),m,n) \\ & - \tilde{\lambda}_1 V(x(t),t,\bar{r}(t)),\ t \in [\bar{t}_{2l-2},\bar{t}_{2l-1}),\end{aligned} \tag{3.81}$$

and

$$
\begin{aligned}
|x(t)| \geq \bar{\chi}(\|u\|_{[0,\infty)}) \Rightarrow \mathfrak{L}V(x(t), y_1(t), t, \bar{r}(t)) \\
&\leq \lambda_{21} \min_{m\in\mathscr{S}} V(y_1(t), t - d_{1i_l}(t), m, n) \\
&+ \tilde{\lambda}_2 V(x(t), t, \bar{r}(t)),\ t \in [\bar{t}_{2l-1}, \bar{t}_{2l}),
\end{aligned}
\tag{3.82}
$$

for any $l \geq 0$, where $\tilde{\lambda}_1 = \lambda_1 - \lambda_0 > 0$, $\tilde{\lambda}_2 = \lambda_2 + \lambda_0$, and $\bar{\chi}(s) = \lambda_0^{-1}\alpha_1^{-1} \circ \chi(s)$.

Clearly, $\bar{\chi}(\cdot) \in \mathscr{K}$. By using Fatou's lemma, we have

$$
\begin{aligned}
|x(t)| \geq \bar{\chi}(\|u\|_{[0,\infty)}) \Rightarrow E\{\mathfrak{L}V(x(t), y_1(t), t, \bar{r}(t))\} \\
&\leq -\bar{\lambda}_1 E\{V(x(t), t, \bar{r}(t))\},\ t \in [\bar{t}_{2l-2}, \bar{t}_{2l-1}),
\end{aligned}
$$

and

$$
\begin{aligned}
|x(t)| \geq \bar{\chi}(\|u\|_{[0,\infty)}) \Rightarrow E\{\mathfrak{L}V(x(t), y_1(t), t, \bar{r}(t))\} \\
&\leq \bar{\lambda}_2 E\{V(x(t), t, \bar{r}(t))\},\ t \in [\bar{t}_{2l-1}, \bar{t}_{2l}),
\end{aligned}
$$

whenever (3.19) holds.

Thus, all the conditions in the Theorem 3.2 are satisfied, which means system (3.76) is SISS.

Corollary 3.6 *Under the hypotheses of Corollary 3.5, system (3.76) is also α_1-ISSiM. Specially, if $\alpha_1(s) = c_1 s^p$, $\alpha_2(s) = c_2 s^p$, where c_1 and c_2 are positive numbers, system (3.76) is pth moment ISS.*

From the definitions of SISS and pth moment ISS, a SISS/pth moment ISS system is GASiP/pth moment stable if the input $u = 0$. A pth moment ISS system is also SISS. Therefore, in what follows we give only the conditions of the pth moment ISS for a class of asynchronous HSDSs.

Consider the following system

$$
\begin{aligned}
\mathrm{d}x(t) = {} & [A(r(t))x(t) + B(r(t))\nu(t) + f(t, x(t - d_1(t, r(t))), r(t))]\mathrm{d}t \\
& + [C(r(t))x(t) + g(t, x(t - d_1(t, r(t))), r(t))]\mathrm{d}B(t),
\end{aligned}
\tag{3.83}
$$

where $x(t) \in \mathbb{R}^n$, $\nu(t) \in \mathscr{L}_\infty^l$. (For such system, the linear case with constant delay has been discussed in [23] and the references therein.)

Assume that

$$
\begin{aligned}
|f(t, x(t - d_1(t, r(t))), r(t))| \leq \|U_1(r(t))\| |x(t - d_1(t, r(t)))| \\
|g(t, x(t - d_1(t, r(t))), r(t))| \leq \|U_2(r(t))\| |x(t - d_1(t, r(t)))|
\end{aligned}
$$

The mode-dependent controller is designed as

$$v(t) = K(r'(t))x(t) + u(t), \tag{3.84}$$

where $u(t)$ is the reference input.

For convenience, when $r(t) = i$, for any operate h, let h_i denote $h(i)$, and $y_1(t) = x(t - d_{1i}(t))$. Then, the closed-loop system is

$$\begin{aligned} \mathrm{d}x(t) = [&A_i x(t) + B_i K_j x(t) + B_i u(t) \\ &+ f_i(t, y_1(t))]\mathrm{d}t + [C_i x(t) + g_i(t, y_1(t))]\mathrm{d}B(t). \end{aligned} \tag{3.85}$$

Taking $V(x(t), \bar{r}(t)) = x^T(t)P(\bar{r}(t))x(t)$, where $P(\bar{r}(t)) = P^T(\bar{r}(t)) > 0$, if for some $\varepsilon_i > 0$, $i = 1, 2, 3$, such that

$$\begin{bmatrix} \Sigma_{111} & \Sigma_{112} & \Sigma_{113} \\ * & \Sigma_{122} & \Sigma_{123} \\ * & * & \Sigma_{133} \end{bmatrix} < 0, \tag{3.86}$$

$$\begin{bmatrix} \Sigma_{211} & X_{ii} \\ * & -\lambda_{11} X_{ii} \end{bmatrix} < 0, \tag{3.87}$$

$$\begin{bmatrix} \Sigma_{311} & \Sigma_{312} & \Sigma_{313} \\ * & \Sigma_{322} & \Sigma_{323} \\ * & * & \Sigma_{333} \end{bmatrix} < 0, \tag{3.88}$$

$$\begin{bmatrix} \Sigma_{411} & X_{ij} \\ * & -\lambda_{21} X_{ij} \end{bmatrix} < 0, \tag{3.89}$$

where $X_{ii} = P_{ii}^{-1}$, $X_{ij} = P_{ij}^{-1}$, $P_{ii} < \beta_1 I$ and $P_{ij} < \beta_2 I$, $\Sigma_{111} = \frac{1}{\beta_2 \pi_{ii}} I$, $\Sigma_{112} = X_{ii}$, $\Sigma_{113} = 0$, $\Sigma_{122} = -\frac{1}{1+\varepsilon_3} X_{ii}$, $\Sigma_{123} = C_i X_{ii}$, $\Sigma_{311} = -\frac{1}{\pi_{ji}^0} X_{ii}$, $\Sigma_{312} = X_{ij}$, $\Sigma_{313} = 0$, $\Sigma_{322} = -\frac{1}{1+\varepsilon_3} X_{ij}$, $\Sigma_{323} = C_i X_{ij}$, $\Sigma_{133} = X_{ii} A_i^T + A_i X_{ii} + 2B_i Y_{ii} + \pi_{ii} X_{ii} + \varepsilon_1 B_i B_i^T + \varepsilon_2 I$,

$$\begin{aligned} \Sigma_{211} &= -(\varepsilon_2^{-1} \|U_{1i}\|^2 I + (1 + \varepsilon_3^{-1})\beta_1 \|U_{2i}\|^2 I)^{-1} I + \lambda_1 X_{ii} \\ \Sigma_{333} &= X_{ij} A_i^T + A_i X_{ij} + 2B_i K_j X_{ij} - \pi_{ji}^0 X_{ij} + \varepsilon_1 B_i B_i^T + \varepsilon_2 I - \lambda_2 X_{ij} \\ \Sigma_{411} &= -(\varepsilon_2^{-1} \|U_{1i}\|^2 I + (1 + \varepsilon_3^{-1})\beta_2 \|U_{2i}\|^2 I)^{-1} I \end{aligned}$$

Let $\chi(s) = \varepsilon_1^{-1} s^2$, and if there exists $\mu \geq 1$, $q > 1$, $\lambda_0 > 0$, such that (3.21), (3.79), (3.80) hold, where $\bar{\lambda}_1 = \lambda_1 - q\lambda_{11} - \lambda_0 > 0$, $\bar{\lambda}_2 = \lambda_2 + q\lambda_{21} + \lambda_0 > 0$, $\hat{\lambda}_1 \in (0, \bar{\lambda}_1)$ and $\hat{\lambda}_2 \in (\bar{\lambda}_2, \infty)$.

Now we show that system (3.85) is 2nd moment ISS by use of Corollary 3.6. Let $V(x(t), \bar{r}(t)) = x^T(t)P(\bar{r}(t))x(t)$, where $P(\bar{r}(t)) = P^T(\bar{r}(t)) > 0$. For any $i, j \in \mathscr{S}$, there exist $\beta_1 > 0$ and $\beta_2 > 0$ such that $P_{ii} < \beta_1 I$ and $P_{ij} < \beta_2 I$, where I is an identity matrix with an appropriate dimension. Since $P_{ij} = P_{ij}^T > 0$, there exists a low-triangular matrix L_{ij} such that $P_{ij} = L_{ij}L_{ij}^T$. From [20], $HFE + E^T F^T H^T \leq \varepsilon HH^T + \varepsilon^{-1}E^T E, \forall\varepsilon > 0$, when $FF^T \leq I$. Then, for any time-interval $[\bar{t}_{2l-1}, \bar{t}_{2l})$, if there exists $\lambda_2 > 0$, $\lambda_{21} > 0$,

$$\begin{aligned}
&\mathfrak{L}V(x(t), y_1(t), i, j)\\
&\leq x^T(t)[A_i^T P_{ij} + P_{ij}A_i + C_i^T P_{ij}C_i + 2P_{ij}B_iK_j\\
&+\pi_{ji}^0 P_{ii} - \pi_{ji}^0 P_{ij}]x(t) + 2x^T(t)P_{ij}B_iu(t)\\
&+2x^T(t)P_{ij}f_i(t, y_1(t)) + 2x^T(t)C_i^T P_{ij}g_i(t, y_1(t))\\
&+g_i^T(t, y_1(t))P_{ij}g_i(t, y_1(t))\\
&\leq x^T(t)[A_i^T P_{ij} + P_{ij}A_i + (1+\varepsilon_3)C_i^T P_{ij}C_i\\
&+2P_{ij}B_iK_j + \pi_{ji}^0 P_{ii} - \pi_{ji}^0 P_{ij} + \varepsilon_1 P_{ij}B_iB_i^T P_{ij}\\
&+\varepsilon_2 P_{ij}P_{ij}]x(t) + \varepsilon_1^{-1}u^T(t)u(t)\\
&+[\varepsilon_2^{-1}\|U_{1i}\|^2 + (1+\varepsilon_3^{-1})\beta_2\|U_{2i}\|^2]y_1^T(t)y_1(t)\\
&\leq \lambda_2 x^T(t)P_{ij}x(t) + \lambda_{21}y_1^T(t)P_{ij}y_1(t) + \varepsilon_1^{-1}|u(t)|^2,
\end{aligned}$$

for any $\varepsilon_i > 0$, $i = 1, 2, 3, 4$.

Similarly, when $t \in [\bar{t}_{2l}, \bar{t}_{2l+1})$, if there also exists $\lambda_1 > 0$, $\lambda_{11} > 0$, such that

$$\begin{aligned}
&\mathfrak{L}V(x(t), y_1(t), y_2(t), i, i)\\
&\leq x^T(t)[A_i^T P_{ii} + P_{ii}A_i + (1+\varepsilon_3)C_i^T P_{ii}C_i\\
&+2P_{ii}B_iK_i + \pi_{ii}P_{ii} - \pi_{ii}\beta_2 I + \varepsilon_1 P_{ii}B_iB_i^T P_{ii}\\
&+\varepsilon_2 P_{ii}P_{ii}]x(t) + \varepsilon_1^{-1}u^T(t)u(t)\\
&+[\varepsilon_2^{-1}\|U_{1i}\|^2 + (1+\varepsilon_3^{-1})\beta_1\|U_{2i}\|^2]y_1^T(t)y_1(t)\\
&\leq -\lambda_1 x^T(t)P_{ii}x(t) + \lambda_{11}y_1^T(t)P_{ii}y_1(t) + \varepsilon_1^{-1}|u(t)|^2.
\end{aligned}$$

Then,

$$\begin{aligned}
&A_i^T P_{ii} + P_{ii}A_i + (1+\varepsilon_3)C_i^T P_{ii}C_i + 2P_{ii}B_iK_i\\
&+\pi_{ii}P_{ii} - \pi_{ii}\beta_2 I + \varepsilon_1 P_{ii}B_iB_i^T P_{ii} + \varepsilon_2 P_{ii}P_{ii} + \lambda_1 P_{ii} < 0, \qquad (3.90)\\
&A_i^T P_{ij} + P_{ij}A_i + (1+\varepsilon_3)C_i^T P_{ij}C_i + 2P_{ij}B_iK_j\\
&+\pi_{ji}^0 P_{ii} - \pi_{ji}^0 P_{ij} + \varepsilon_1 P_{ij}B_iB_i^T P_{ij} + \varepsilon_2 P_{ij}P_{ij} - \lambda_2 P_{ij} < 0, \qquad (3.91)\\
&\varepsilon_2^{-1}\|U_{1i}\|^2 I + (1+\varepsilon_3^{-1})\beta_1\|U_{2i}\|^2 I - \lambda_{11}P_{ii} < 0, \qquad (3.92)\\
&\varepsilon_2^{-1}\|U_{1i}\|^2 I + (1+\varepsilon_3^{-1})\beta_2\|U_{2i}\|^2 I - \lambda_{21}P_{ij} < 0. \qquad (3.93)
\end{aligned}$$

Using P_{ii}^{-1} to pre- and post- multiply the left term of Eqs. (3.90) and (3.92) respectively yields (3.86) and (3.87) hold. Similarly, using P_{ij}^{-1} to pre- and post- multiply the left term of Eqs. (3.91) and (3.93) respectively yields (3.88) and (3.89) hold.

Thus, when let $\chi(s) = \varepsilon_1^{-1} s^2$, and if there exists $\mu \geq 1$, $q > 1$, $\lambda_0 > 0$, such that (3.53), (3.79), (3.80) and (3.86)–(3.89) hold. Then, according to Schur's complement and Corollary 3.6, system (3.85) is 2nd moment ISS.

For the stability analysis of given system (3.83) with asynchronous controller (3.84), we first obtain μ, λ_1, λ_2, λ_{11} and λ_{21}, which meet the conditions of Corollary 3.6. If there exist ε_1, ε_2, ε_3, β_1 and β_2, such that (3.86) and (3.87) hold, then we can obtain P_{ii} and the candidate controllers gains K_i, where $i \in \mathscr{S}$. To verify the effectiveness of the candidate controllers, we need to solve (3.88), (3.89) and (3.21). If a feasible solution exists, then one can obtain P_{ij} and the admissible controllers gains, where $i, j \in \mathscr{S}$, $j \neq i$.

Example 3.1 To demonstrate the effectiveness, we choose the parameters in system (3.85) as $A_1 = [1.5, 1.5; 0, -3]$, $A_2 = [-0.5, 10; 15, 2.5]$, $B_1 = [-1, 2; 0, -1]$, $B_2 = [-2, 1; 0, 2]$, $C_1 = [0.1, 0; 0, 0.1]$, $C_2 = [0.2, 0; 0.1, 0.2]$, and

$$
\begin{aligned}
f_1(t, y_1(t)) &= \begin{bmatrix} 0.1\cos(t) & 0.1 \\ 0 & -0.1\sin(t) \end{bmatrix} y_1(t), \\
f_2(t, y_1(t)) &= \begin{bmatrix} 0.1(\cos(t))^2 & 0 \\ 0 & 0.1\sin(t) \end{bmatrix} y_1(t), \\
g_1(t, y_1(t)) &= \begin{bmatrix} 0.1\cos(t) & 0 \\ 0 & -0.1\sin(t) \end{bmatrix} y_1(t), \\
g_2(t, y_1(t)) &= \begin{bmatrix} 0.1\cos(t) & 0 \\ 0.1 & 0.1(\sin(t))^2 \end{bmatrix} y_1(t).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&|f_1(t, y_1(t))| \leq \|U_{11}\||y_1(t)|, \quad |f_2(t, y_1(t))| \leq \|U_{12}\||y_1(t)|, \\
&|g_1(t, y_1(t))| \leq \|U_{21}\||y_1(t)|, \quad |g_2(t, y_1(t))| \leq \|U_{22}\||y_1(t)|,
\end{aligned}
$$

where $U_{11} = [0.1, 0.1; 0, -0.1]$, $U_{12} = [0.1, 0; 0, 0.1]$, $U_{21} = [0.1, 0; 0, -0.1]$, $U_{22} = [0.1, 0; 0.1, 0.1]$, and $d_{11}(t) = 0.05\cos(2t)$, $d_{12}(t) = 0.07\sin(t)$, $d_{21}(t) = 0.06\sin(t)$, $d_{22}(t) = 0.08\cos(t)$, $\tau = 0.08$. Suppose that $d = 0.2$, $\Pi = [-0.01, 0.01; 0.01, -0.01]$ and $\Pi^0 = [-70, 70; 50, -50]$.

According to above analysis, we choose $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.6$, $\varepsilon_3 = 1.8$, $\lambda_1 = 20$, $\lambda_2 = 18$, $\lambda_{11} = 1.5$, $\lambda_{21} = 1.5$, $\beta_1 = 8$, $\beta_2 = 3$ and $\mu = 1.5$. There exists $\lambda_0 = 0.01$, $q = 2$, such that $\bar{\lambda}_1 = 16.99$, $\bar{\lambda}_2 = 21.01$.

Further, there exists $\hat{\lambda}_1 = 5.097 \in (0, 16.99)$ and $\hat{\lambda}_2 = 21.031 \in (21.01, \infty)$, such that $2 = q > e^{\hat{\lambda}_1 \tau} = 1.5034$. It's not difficult to verify that (3.80) holds with those parameters and $\varsigma = 0.99$, $\bar{\pi} = \tilde{\pi} = 0.01$.

By solving (3.21), (3.86)–(3.89), one can obtain that

$$P_{11} = \begin{bmatrix} 0.1854 & 0 \\ 0 & 0.1854 \end{bmatrix}, \quad P_{12} = \begin{bmatrix} 0.2670 & -0.0011 \\ -0.0011 & 0.2703 \end{bmatrix},$$
$$P_{21} = \begin{bmatrix} 0.1943 & 0.0446 \\ 0.0446 & 0.5675 \end{bmatrix}, \quad P_{22} = \begin{bmatrix} 0.3826 & 0.0004 \\ 0.0004 & 0.3823 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} 14.0981 & 20.9377 \\ -0.0706 & 9.7021 \end{bmatrix}, K_2 = \begin{bmatrix} 8.3710 & 9.5497 \\ 1.4964 & -9.0793 \end{bmatrix}.$$

The simulation results are shown in Figs. 3.2, 3.3, 3.4, 3.5 and 3.6. Among them, Fig. 3.2 shows the Markovian switching signal which includes the real switching signal and the detected switching signal with non-zero detection delay. The detected switching signal also includes both the case which $r'(t)$ satisfies the conditions of the Corollary 3.6 and the case which $r'(t)$ doesn't satisfy the conditions of the Corollary 3.6.

In the later case, the maximum detection delay is larger than 0.3, then we have $\mu^2 \bar{\pi} e^{0.3(\hat{\lambda}_1+\hat{\lambda}_2)} - \tilde{\pi} = 57.0534 > \hat{\lambda}_1$. Moreover, in order to distinguish the $r'(t)$, we let value 1.1 and 2.1 to express the mode 1 and mode 2 of $r'(t)$ which doesn't satisfy the conditions. Figure 3.3a shows the curve of Brownian motion $w(t)$; Fig. 3.3b shows the state trajectories under control input $v(t) \equiv 0$, with initial data $x_0 = [3, -1.5]$. Obviously, system (3.83) under $v(t) \equiv 0$ is unstable, i.e., the open-loop system is unstable. Figures 3.4, 3.5 and 3.6 show the stability of the closed-loop system, also with initial data $x_0 = [3, -1.5]$. Among them, Figs. 3.4a, 3.5a and 3.6a show the stability under the strictly synchronous controller, where the reference input $u(t)$, respectively, equals to $[0, 0]^T$, $[3, 3]^T$ and $[3e^{-0.4t}, 5e^{-0.7t}]^T$. The so-called strictly synchronous controller means that the controller in (3.84) relies not on the

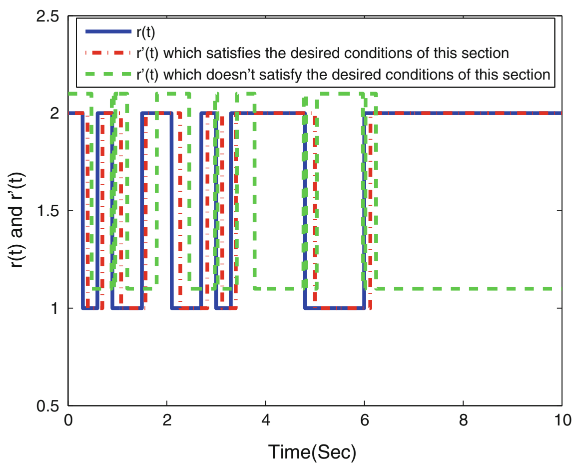


Fig. 3.2 The switching signal $r(t)$ and the detected $r'(t)$

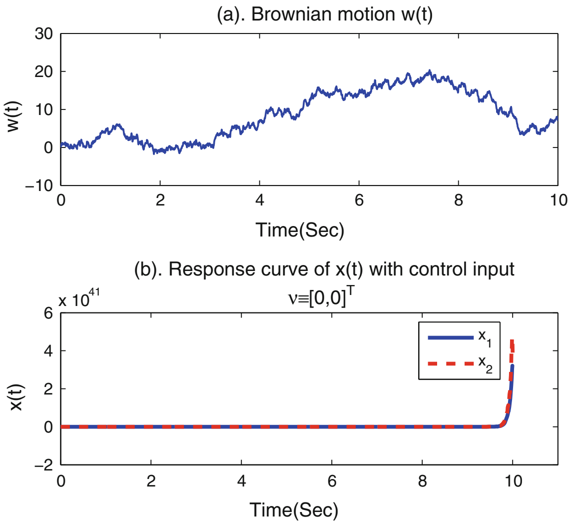


Fig. 3.3 Response curve of $w(t)$ and $x(t)$

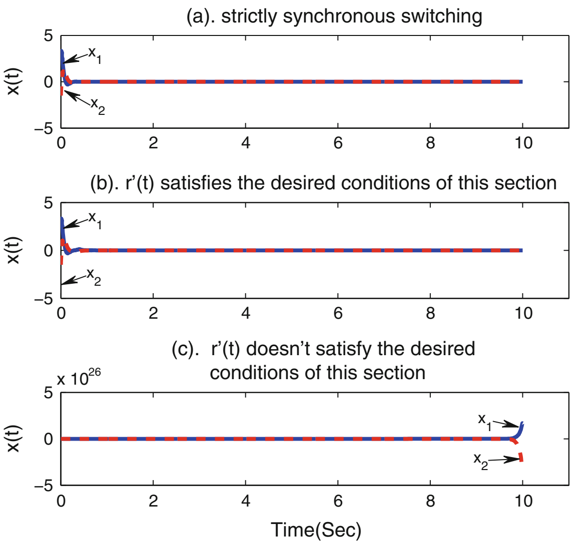


Fig. 3.4 Response curve of $x(t)$ with reference input $u \equiv [0, 0]^T$

the detected switching signal $r'(t)$ but on actual $r(t)$. It can be inferred from them that the system under synchronous switching is stable. On the other hand, Figs. 3.4b, 3.5b and 3.6b show the stability under $r'(t)$ which satisfies the conditions of Corollary 3.6.

Obviously, the asymptotic stability and the input-to-state stability under $r'(t)$ which satisfies the conditions can be guaranteed. But compared with Figs. 3.4a, 3.5a and 3.6a, one can see that the mismatched controller which caused by the non-zero detection delay has a great influence on the performance of the system. And moreover, when $r'(t)$ doesn't satisfy the conditions of Corollary 3.6, the system is unstable, as

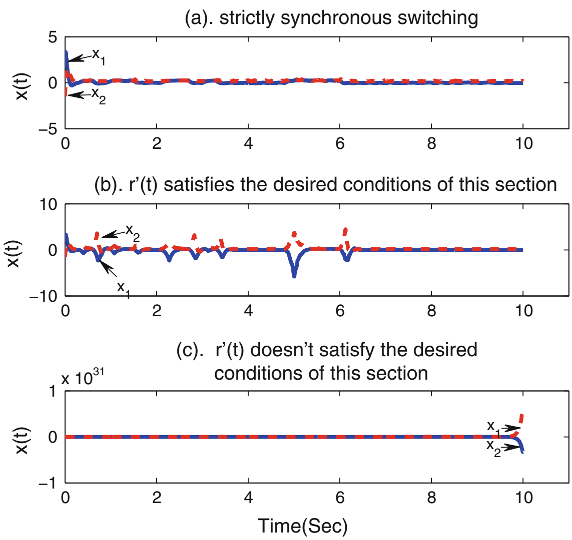


Fig. 3.5 Response curve of $x(t)$ with reference input $u \equiv [3, 3]^T$

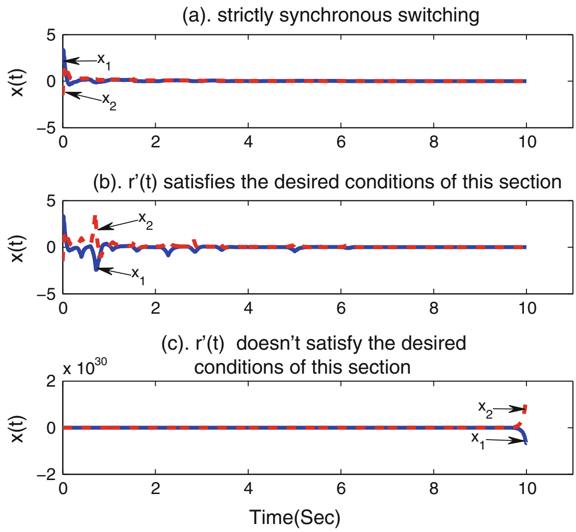


Fig. 3.6 Response curve of $x(t)$ with reference input $u = [3e^{-0.4t}, 5e^{-0.7t}]^T$

shown in Figs. 3.4c, 3.5c and 3.6c, which corresponds to Figs. 3.4b, 3.5b and 3.6b, respectively. In addition, from Fig. 3.4a, b, we can see that the closed-loop system (3.85) is asymptotically stable, which is in accordance with the assertion that an ISS system is necessarily asymptotically stable. In Fig. 3.5a, b, due to the effect of reference input u, the state $x(t)$ will not converge to zero. But, it still remains bounded. In Fig. 3.6a, b, since $|u(t)| \to 0$ as $t \to \infty$, system (3.85) is asymptotically stable, which is also in accordance with Remark 3.1 in [2].

3.4.2 Extended Asynchronous Switching

In this section the general Razumikhin-type theorems established in the previous section will be extended to deal with the input-to-state stability of switched stochastic nonlinear delay system (SSNLDS).

For a simulation purpose, consider a special class of switched stochastic perturbed system

$$\mathrm{d}x = [A_r x + B_r v]\mathrm{d}t + g(t, x(t - d(t)), r)\mathrm{d}B, \tag{3.94}$$

where $g : \mathbb{R}_+ \times \mathbb{R}^n \times \mathscr{S} \rightarrow \mathbb{R}^n$ is unknown nonlinear function satisfying the local Lipschitz condition and the linear growth condition, and $\|g(t, x(t - d(t)), i)\|_2 \leq \|U_i x(t - d(t))\|_2$, $\|\cdot\|_2$ denotes the 2-norm, for any $i \in \mathscr{S}$. U_i is known real constant matrix, and $0 \leq d(t) \leq \tau$. Design $v(t) = K_{r'(t)}x(t) + u(t)$. Then, the closed-loop system is

$$\mathrm{d}x = [A_r x + B_r K_{r'} x + B_r u]\mathrm{d}t + g(t, x(t - d(t)), r)\mathrm{d}B. \tag{3.95}$$

From Corollary 3.4, one has the following corollary.

Corollary 3.7 *System (3.95) is 2nd moment ISS for all* $\tau^* > \frac{\ln(\mu M)}{\hat{\lambda}_1}$, *where*

$$M = (1 + \mu)[\frac{-\bar{\pi}^0}{\hat{\lambda}_1 + \hat{\lambda}_2 - \underline{\pi}^0} + (\frac{-\bar{\pi}^0}{\hat{\lambda}_1 + \hat{\lambda}_2 - \underline{\pi}^0})^{\frac{1}{2}}] \times e^{[(\mu^2 - \frac{\bar{\pi}^0}{\hat{\lambda}_1 + \hat{\lambda}_2 - \underline{\pi}^0})(N-1) - 2]\tilde{\pi}^1 \varsigma}, \tag{3.96}$$

if for all $i, j \in \mathscr{S}$, *there exist* $X_{ij} = X_{ij}^T > 0$, $\lambda_1 > 0$, $\lambda_2 \geq 0$, $\lambda_{10} \geq 0$, $\lambda_{20} \geq 0$ *such that (3.97)–(3.100) hold, i.e.,*

$$\begin{bmatrix} \Pi_1 & X_{ii} & X_{ii} & \cdots & X_{ii} & X_{ii} & \cdots & X_{ii} & X_{ii} \\ * & -\frac{1}{\pi_{i1}^1}X_{i1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ * & * & -\frac{1}{\pi_{i2}^1}X_{i2} & \cdots & 0 & 0 & \cdots & 0 & 0 \\ * & * & * & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & * & -\frac{1}{\pi_{i(i-1)}^1}X_{i(i-1)} & 0 & \cdots & 0 & 0 \\ * & * & * & * & * & -\frac{1}{\pi_{i(i+1)}^1}X_{i(i+1)} & \cdots & 0 & 0 \\ * & * & * & * & * & * & \ddots & \vdots & \vdots \\ * & * & * & * & * & * & * & -\frac{1}{\pi_{iN}^1}X_{iN} & 0 \\ * & * & * & * & * & * & * & * & -Q^{-1} \end{bmatrix} \leq 0, \tag{3.97}$$

$$\begin{bmatrix} -\lambda_{10}X_{ii} & X_{ii}U_i^T \\ * & -\frac{2}{\beta_1}I \end{bmatrix} \leq 0, \tag{3.98}$$

$$\begin{bmatrix} \Pi_2 & X_{ij} & X_{ij} \\ * & -\frac{1}{\pi_{ji}^0}X_{ii} & 0 \\ * & * & -Q^{-1} \end{bmatrix} \leq 0, \tag{3.99}$$

$$\begin{bmatrix} -\lambda_{20}X_{ij} & X_{ij}U_i^T \\ * & -\frac{2}{\beta_2}I \end{bmatrix} \leq 0, \tag{3.100}$$

where $\Pi_1 = A_iX_{ii} + X_{ii}A_i^T + Y_{ii}^TB_i^T + B_iY_{ii} + \varepsilon_1B_iB_i^T + (\lambda_1 + \pi_{ii}^1)X_{ii}$, $\Pi_2 = X_{ij}A_i^T + A_iX_{ij} + Y_{ij}^TB_i^T + B_iY_{ij} + \varepsilon_2B_iB_i^T - (\lambda_2 + \pi_{ji}^0)X_{ij}$, $\varepsilon_1, \varepsilon_2 > 0$*; there exist* $q > 1$*, such that* $\bar{\lambda}_1 = \lambda_1 - q\lambda_{10} > 0$*, and*

$$e^{\hat{\lambda}_1\tau} < q, \tag{3.101}$$

$$\hat{\lambda}_1 + \hat{\lambda}_2 - \underline{\pi}^0 < 0, \tag{3.102}$$

where $\bar{\lambda}_2 = \lambda_2 + q\lambda_{20}$*,* $\hat{\lambda}_1 \in (0, \bar{\lambda}_1)$ *and* $\hat{\lambda}_2 \in (\bar{\lambda}_2, \infty)$.

Proof Take $V(x(t), t, i, j) = x^T(t)P_{ij}x(t)$, $P_{ij} = P_{ij}^T > 0$, for any $i, j \in \mathscr{S}$. We assume that there exist $\beta_1 > 0$ and $\beta_2 > 0$ such that $P_{ii} < \beta_1 I$ and $P_{ij} < \beta_2 I$, where I is an identity matrix with appropriate dimension.

When $t \in T_s(t_l, t_{l+1})$, the system in (3.95) can be written as

$$\mathrm{d}x = [(A_i + B_iK_i)x + B_iu]\mathrm{d}t + g(t, x(t-d), i)\mathrm{d}B. \tag{3.103}$$

Then,

$$\begin{aligned}
\mathfrak{L}V(x, y, t, i, i) &= 2x^TP_{ii}[(A_i + B_iK_i)x + B_iu] \\
&\quad + \frac{1}{2}g^T(t, y, i)P_{ii}g(t, y, i) + \sum_{k=1}^{N}\pi_{ik}^1x^TP_{ik}x \\
&\leq x^T[(A_i + B_iK_i)^TP_{ii} + P_{ii}(A_i + B_iK_i) + \varepsilon_1P_{ii}B_iB_i^TP_{ii} \\
&\quad + \sum_{k=1}^{N}\pi_{ik}^1P_{ik}]x + \varepsilon_1^{-1}u^Tu + \frac{1}{2}\beta_1g^T(t, y, i)g(t, y, i) \\
&\leq x^T[(A_i + B_iK_i)^TP_{ii} + P_{ii}(A_i + B_iK_i) + \varepsilon_1^{-1}\|u\|_2^2 \\
&\quad + \varepsilon_1P_{ii}B_iB_i^TP_{ii} + \sum_{k=1}^{N}\pi_{ik}^1P_{ik}]x + \frac{1}{2}\beta_1y^TU_i^TU_iy
\end{aligned}$$

for some $\varepsilon_1 > 0$, where $y(t) = x(t - d(t))$, by considering the fact that $HFE + E^TF^TH^T \leq \varepsilon HH^T + \varepsilon^{-1}E^TE$ where $\varepsilon > 0$, $FF^T \leq I$.

When $t \in T_a(t_l, t_{l+1})$, the system in (3.95) can be written as

$$\mathrm{d}x = [(A_i + B_i K_j)x + B_i u]\mathrm{d}t + g(t, x(t-d), i)\mathrm{d}B, \tag{3.104}$$

where $i, j \in \mathscr{S}$, and $i \neq j$.

Similarly, for some $\varepsilon_2 > 0$, we have

$$\begin{aligned}\mathfrak{L}V(x, y, t, i, j) \leq x^T[&(A_i + B_i K_j)^T P_{ij} + P_{ij}(A_i + B_i K_j) + \varepsilon_2 P_{ij} B_i B_i^T P_{ij} \\ &+ \pi_{ji}^0 (P_{ii} - P_{ij})]x + \frac{1}{2}\beta_2 y^T U_i^T U_i y + \varepsilon_2^{-1}\|u\|_2^2.\end{aligned}$$

For any non-negative definite matrix Q, we have

$$|x| \geq \sqrt{\frac{1}{\varepsilon \lambda_{min}(Q)}}\|u\|_2 \Rightarrow \begin{cases} \mathfrak{L}V(x, y, t, i, i) \leq x^T \Phi_{ii} x + \frac{1}{2}\beta_1 y^T U_i^T U_i y, \\ \mathfrak{L}V(x, y, t, i, j) \leq x^T \Phi_{ij} x + \frac{1}{2}\beta_2 y^T U_i^T U_i y, \end{cases}$$

where $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, $\lambda_{min}(Q)$ denotes the minimal eigenvalue of matrix Q,

$$\begin{aligned}\Phi_{ii} &= (A_i + B_i K_i)^T P_{ii} + P_{ii}(A_i + B_i K_i) + \varepsilon_1 P_{ii} B_i B_i^T P_{ii} + \sum_{k=1}^{N} \pi_{ik}^1 P_{ik} + Q, \\ \Phi_{ij} &= (A_i + B_i K_j)^T P_{ij} + P_{ij}(A_i + B_i K_j) + \varepsilon_2 P_{ij} B_i B_i^T P_{ij} + \pi_{ji}^0 (P_{ii} - P_{ij}) + Q.\end{aligned}$$

Further, if

$$\mathfrak{L}V(x, y, t, i, i) \leq x^T \Phi_{ii} x + \frac{1}{2}\beta_1 y^T U_i^T U_i y \leq -\lambda_1 x^T P_{ii} x + \lambda_{10} y^T P_{ii} y, \tag{3.105}$$

and

$$\mathfrak{L}V(x, y, t, i, j) \leq x^T \Phi_{ij} x + \frac{1}{2}\beta_2 y^T U_i^T U_i y \leq \lambda_2 x^T P_{ij} x + \lambda_{20} y^T P_{ij} y, \tag{3.106}$$

and (3.53) and (3.73) hold, then based on Corollary 3.4, the conclusion is obtained.

Moreover, the conditions in (3.105) and (3.106) can be transformed into

$$\begin{aligned}&(A_i + B_i K_i)^T P_{ii} + P_{ii}(A_i + B_i K_i) + \varepsilon_1 P_{ii} B_i B_i^T P_{ii} \\ &+ \sum_{k=1}^{N} \pi_{ik}^1 P_{ik} + Q + \lambda_1 P_{ii} \leq 0,\end{aligned} \tag{3.107}$$

$$\frac{1}{2}\beta_1 U_i^T U_i - \lambda_{10} P_{ii} \leq 0, \tag{3.108}$$

$$(A_i + B_i K_j)^T P_{ij} + P_{ij}(A_i + B_i K_j) + \varepsilon_2 P_{ij} B_i B_i^T P_{ij} \\ + \pi_{ji}^0 (P_{ii} - P_{ij}) + Q - \lambda_2 P_{ij} \leq 0, \tag{3.109}$$

$$\frac{1}{2}\beta_2 U_i^T U_i - \lambda_{20} P_{ij} \leq 0. \tag{3.110}$$

where $i, j \in \mathscr{S}$ and $i \neq j$. Using P_{ii}^{-1} to pre- and post- multiply the left term of Eqs. (3.107) and (3.108) respectively and denoting $X_{ii} = P_{ii}^{-1}$, $X_{ij} = P_{ij}^{-1}$, $Y_{ii} = K_i X_{ii}$ and $Y_{ij} = K_j X_{ij}$ yields (3.97) and (3.98).

Similarly, using P_{ij}^{-1} to pre- and post- multiply the left term of Eqs. (3.109) and (3.110) respectively yields (3.99) and (3.100). It is easy to get that (3.105) and (3.106) hold, if the LMIs (3.97)–(3.100) hold. By taking proper λ_i and $\lambda_{i0}, i = 1, 2$, then there exists q such that (3.101) and (3.102) hold. And further, by solving (3.97)–(3.100) and (3.53), we can get the control gains $K_i, i \in \mathscr{S}$.

Example 3.2 Take the following parameters for system (3.95):

$$A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},\ B_1 = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix},\ U_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 3 & 0 \\ 2 & 3 \end{bmatrix},\ B_2 = \begin{bmatrix} -2 & 1 \\ 0 & 2 \end{bmatrix},\ U_2 = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0 \end{bmatrix},$$

and

$$g(t, \bar{x}(t), 1) = [0.1\cos(t)x_1(t - d(t)), 0]^T,$$
$$g(t, \bar{x}(t), 2) = [0.1\sin(t)x_2(t - d(t)), 0.1x_1(t - d(t))]^T,$$

where $\bar{x}(t) = x(t - d(t))d(t) = 0.2\sin(t)$ with $\tau = 0.2$, and

$$\Pi^0 = \begin{bmatrix} -100 & 100 \\ 80 & -80 \end{bmatrix}, \Pi^1 = \begin{bmatrix} -0.2 & 0.2 \\ 0.2 & -0.2 \end{bmatrix}.$$

Then conditions (3.97), (3.98), (3.99) and (3.100) can be satisfied with $\lambda_1 = 10$, $\lambda_2 = 5.9182$, $\lambda_{10} = 0.1$, $\lambda_{20} = 0.1$, $\varepsilon_1 = 7$, $\varepsilon_2 = 2$, $\mu = 1.38$, $\beta_1 = 1.6956$, $\beta_2 = 1.5955$, $Q = diag[0.1873, 0.1873]$, moreover,

$$P_{11} = \begin{bmatrix} 0.4491 & -0.0001 \\ -0.0001 & 0.4484 \end{bmatrix}, P_{12} = \begin{bmatrix} 0.5877 & -0.0005 \\ -0.0005 & 0.5877 \end{bmatrix},$$
$$P_{21} = \begin{bmatrix} 0.5877 & -0.0005 \\ -0.0005 & 0.5880 \end{bmatrix}, P_{22} = \begin{bmatrix} 0.4483 & -0.0034 \\ -0.0034 & 0.4509 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} 28.9391 & 7.3873 \\ 7.4055 & 9.2980 \end{bmatrix},\quad K_2 = \begin{bmatrix} 5.8994 & -3.4292 \\ -3.3107 & -6.9532 \end{bmatrix}.$$

Take $q = 7.5$, then $\bar{\lambda}_1 = 9.25$ and $\bar{\lambda}_2 = 6.6682$, and further, conditions (3.101) and (3.102) can be satisfied with $\hat{\lambda}_1 = 9.1575$, $\hat{\lambda}_2 = 6.7349$, $\bar{\pi}^0 = 100$, $\tilde{\pi}^1 = 0.2$, $\underline{\pi}^1 = 0.2$ and $\underline{\pi}^0 = 80$. Take $\varsigma = 10$, then $M = 125.0134$ and $\tau^* > 0.5624s$. Then, system (3.95) is 2nd moment ISS with average dwell time $\tau^* = 0.6s$.

The first set of simulations are to verify the necessity of performing the research on extended asynchronous switching. The simulation results are shown in Figs. 3.7, 3.8, 3.9 and 3.10. Among them, Fig. 3.7 shows the response trajectory of the Brownian motion $B(t)$, while Fig. 3.8 gives the state response curves of open-loop system (3.94) with the true switching signal given in Fig. 3.9a, and obviously the open-loop system is unstable. On the other hand, Figs. 3.9b and 3.10b present respectively the state trajectories of closed-loop system under normal asynchronous switching controller and extended asynchronous switching controller (note that, it does not

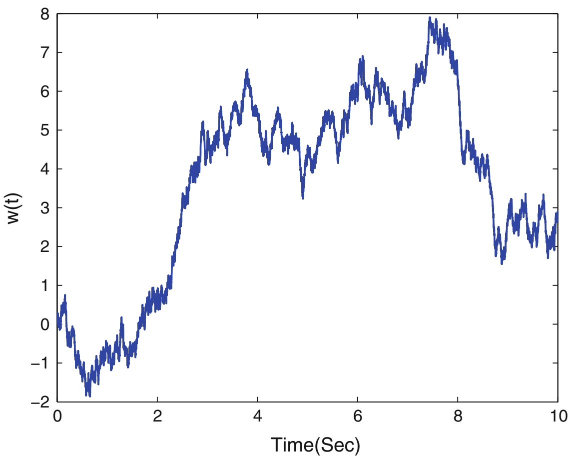


Fig. 3.7 Brownian motion $w(t)$

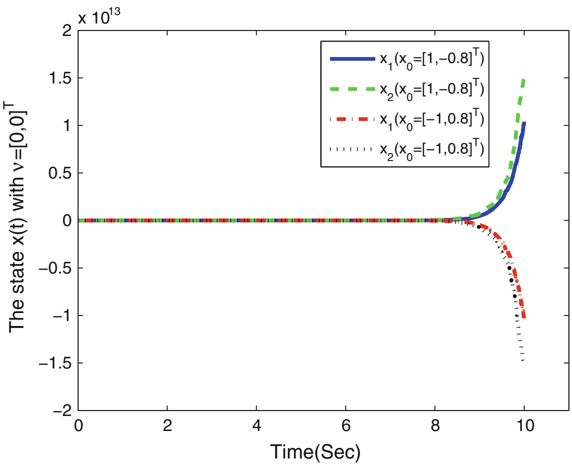


Fig. 3.8 State response curves of the open-loop system

satisfy the conditions of Corollary 3.7, because the average dwell time of the true switching is set to be less than $0.5624s$), where Figs. 3.9a and 3.10a give the switching signals including the true one and the detected one respectively, with the same true switching signal. Comparing the two results, one can find that the false alarm has a great influence on the control performance, which further verifies the necessity and importance of the extended asynchronous switching system.

To demonstrate the effectiveness of the results, the stability under several switching cases are considered, which include the strictly synchronous switching, the desired extended asynchronous switching and the undesired extended asynchronous switching. The simulation results are shown in Figs. 3.11, 3.12, 3.13, 3.14, 3.15, 3.16, 3.17, 3.18, 3.19, 3.20 and 3.21, with the Brownian motion $B(t)$ given in

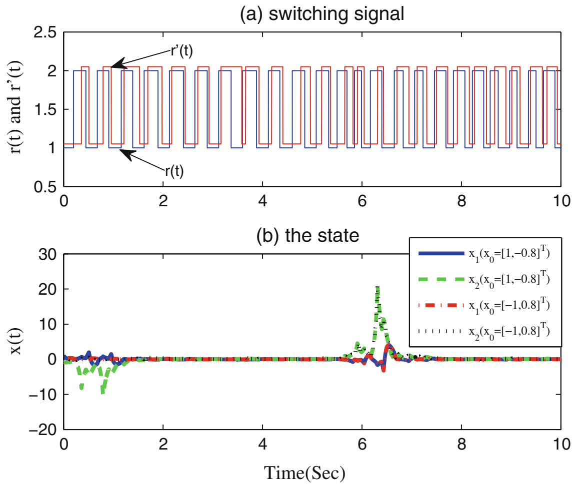


Fig. 3.9 State response curves of the closed-loop system under normal asynchronous switching controller with $u = [2e^{-0.5t}, 4e^{-0.8t}]^T$

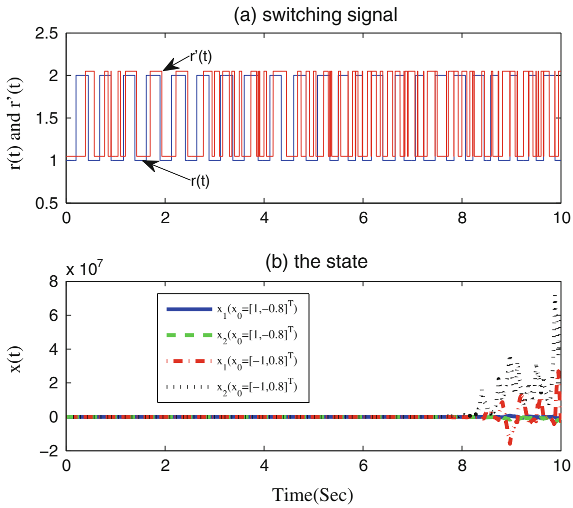


Fig. 3.10 State response curves of the closed-loop system under extended asynchronous switching controller with $u = [2e^{-0.5t}, 4e^{-0.8t}]^T$

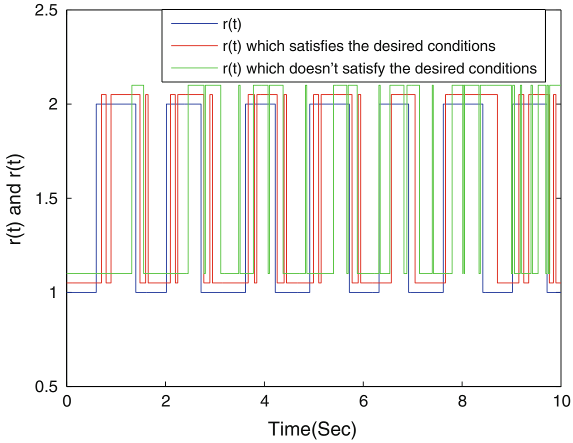


Fig. 3.11 Switching signal $r(t)$ and the detected $r'(t)$

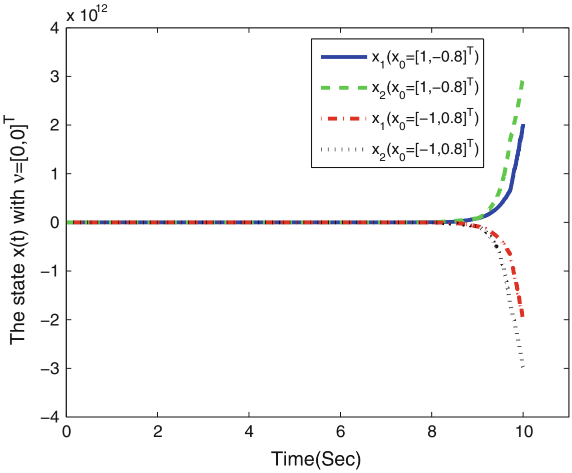


Fig. 3.12 The open-loop state trajectory

Fig. 3.7.Among them, Fig. 3.11 shows the the true switching signal and the detected switching signal in the presence of detection delay and false alarm, where $r(t)$ is the true switching signal with the desired average dwell time $\tau^* = 0.6$, $r'(t)$ with mode 1.05 and mode 2.05 is the detected switching signal which refers to the desired detected signal (the detection parameters are $\Pi^0 = [-100, 100; 80, -80]$, $\Pi^1 = [-0.2, 0.2; 0.2, -0.2]$, all the conditions in Corollary 3.7 are satisfied), while $r'(t)$ with mode 1.1 and mode 2.1 is the undesired detected one (here, we take $\Pi^0 = \Pi^1 = [-10, 10; 10, -10]$, thus (3.102) is not satisfied). Note that both the mode 1.05 (mode 2.05) and mode 1.1 (mode 2.1) are referred the mode 1 (mode 2), and these different values are to make clearer illustration.

Figures 3.13, 3.14, 3.15, 3.16, 3.17, 3.18, 3.19, 3.20 and 3.21 show the response curve of the state trajectories when the average dwell time of the true switching signal

Fig. 3.13 The closed-loop state trajectory with $u = [0, 0]^T$ under strictly synchronous switching controller

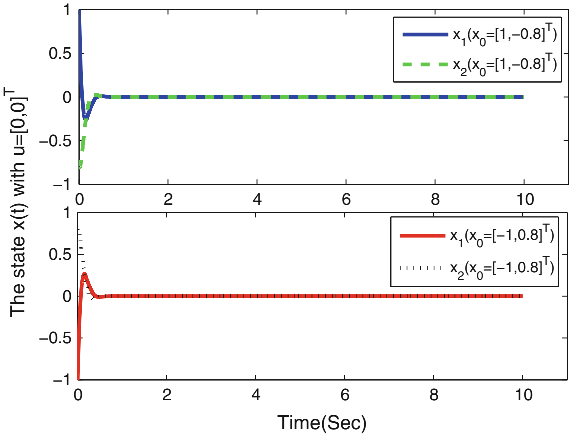


Fig. 3.14 The closed-loop state trajectory with $u = [0.5, 0.5]^T$ under strictly synchronous switching controller

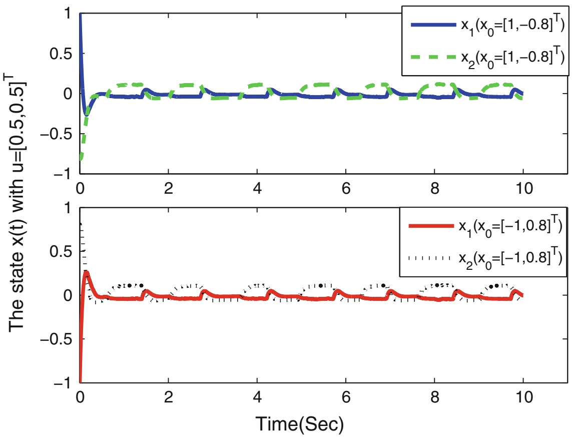
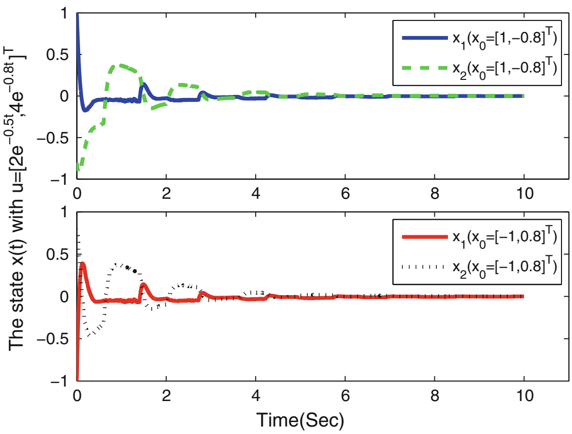


Fig. 3.15 The closed-loop state trajectory with $u = [2e^{-0.5t}, 4e^{-0.8t}]^T$ under strictly synchronous switching controller



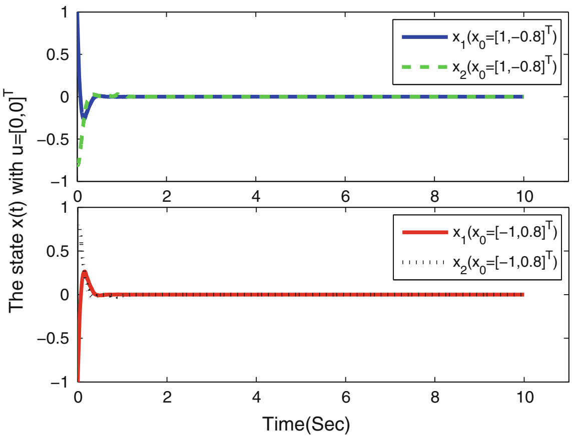


Fig. 3.16 The closed-loop state trajectory with $u = [0, 0]^T$ under extended asynchronous switching controller which satisfies the desired conditions

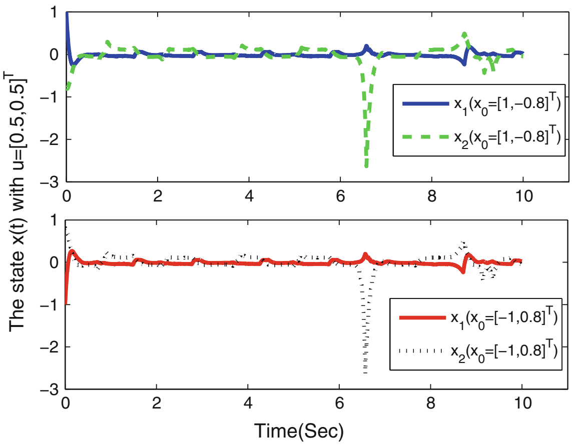


Fig. 3.17 The closed-loop state trajectory with $u = [0.5, 0.5]^T$ under extended asynchronous switching controller which satisfies the desired conditions

is the desired one.Among them, Figs. 3.13, 3.14 and 3.15 show the stability under strictly synchronous switching controller with different reference input respectively.

Similarly, Figs. 3.16, 3.17 and 3.18 show the stability under the desired extended asynchronous switching controller,with reference input $u = [0, 0]^T$, $u = [0.5, 0.5]^T$ and $u = [2e^{-0.5t}, 4e^{-0.8t}]^T$ respectively.

Figures 3.19, 3.20 and 3.21 are performed under the undesired asynchronous switching, with different reference input respectively.

From Figs. 3.13, 3.14 and 3.15, the closed-loop system under strictly synchronous switching controller is stable, in other words, one can claim that the designed controller with considering both detection delay and false alarm (or the designed controller under extended asynchronous controller) are also suitable for the synchronous case. From Figs. 3.16, 3.17 and 3.18, one can find that the designed controller based

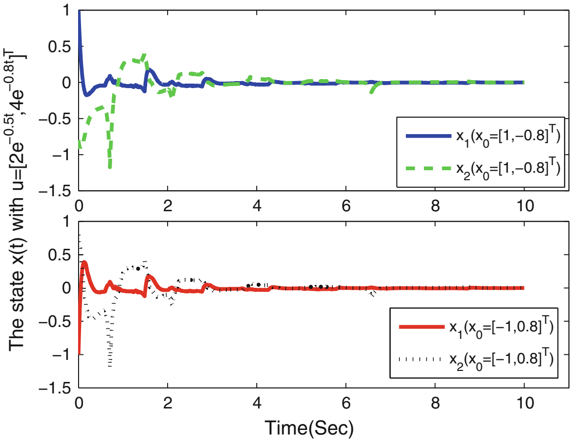


Fig. 3.18 The closed-loop state trajectory with $u = [2e^{-0.5t}, 4e^{-0.8t}]^T$ under extended asynchronous switching controller which satisfies the desired conditions

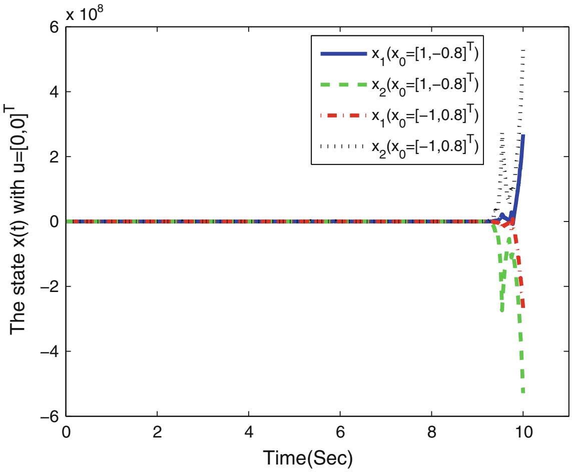


Fig. 3.19 The closed-loop state trajectory with $u = [0, 0]^T$ under extended asynchronous switching controller which doesn't satisfy desired conditions

on the proposed theory can stabilize the switched system with both non-zero detection delay and false alarm in detection. Compared to Figs. 3.13, 3.14 and 3.15, one can also find that the asynchronous phenomenon caused by the non-zero detection delay and false alarm has a great impact on the stability. This point can also be further verified by Figs. 3.19, 3.20 and 3.21. From the results in above three cases, one may claim that the stability of extended asynchronous switching can be guaranteed by a sufficient small mismatched time interval, it is in accordance with Remark 3.9.

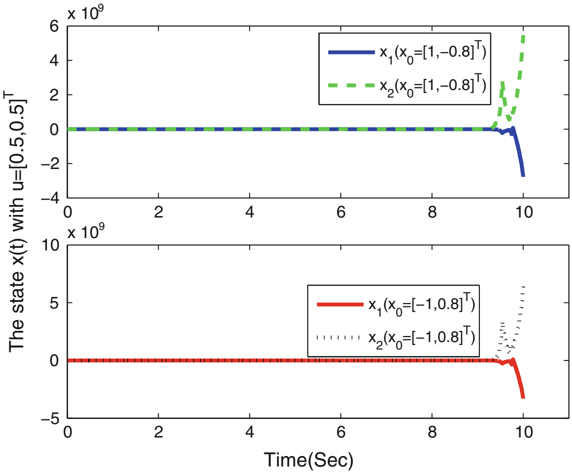


Fig. 3.20 The closed-loop state trajectory with $u = [0.5, 0.5]^T$ under extended asynchronous switching controller that doesn't satisfy desired conditions

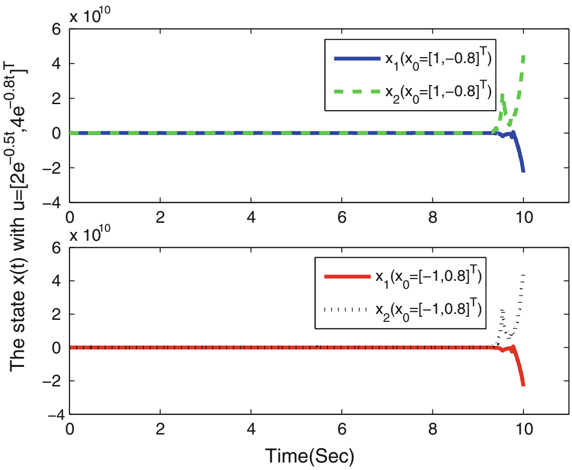


Fig. 3.21 The closed-loop state trajectory with $u = [2e^{-0.5t}, 4e^{-0.8t}]^T$ under extended asynchronous switching controller which doesn't satisfy the desired conditions

3.5 Summary

This chapter, we have studied a class of HSRSs under asynchronous switching and a class of SSNLRS under extended asynchronous switching.

On the one hand, we examine the stability of a class of HSRSs under asynchronous switching, where the detection delay is modeled as a Markovian process. The Razumikhin-type conditions are extended to the interval of asynchronous switching before the matched controller is applied, which allows the Lyapunov functionals to increase during the running time of subsystems. Motivated by asynchronous deterministic switched systems, i.e., the stability of closed-loop systems can be guaranteed by a sufficient large average-dwell time, by considering the properties of Markov process, the conditions of the existence of the admissible asynchronous controller for global asymptotic stability and input-to-state stability are derived. It is shown that the stability of the closed-loop systems can be guaranteed by a sufficient small mode transition rate. The main results have also been applied to a class of hybrid stochastic delay systems, and a numerical example has been provided to demonstrate the effectiveness.

On the other hand, the input-to-state stability of a class of SSNLRS under extended asynchronous switching is also investigated. The switchings of the system modes and the desired mode-dependent controllers are asynchronous due to both detection delays and false alarms, whose feature is different from normal asynchronous switching. Through some simplification, an extended asynchronous switching model is developed. Then, based on Razumikhin-type theorem incorporated with average dwell time approach, the sufficient criteria for asymptotic stability as well as input-to-state stability are proposed. It is shown that the stability of such systems can be guaranteed by a sufficient small mismatched time interval and a sufficient large average dwell time. Finally, the importance and effectiveness of the stability criteria for the extended asynchronous switching system are demonstrated by simulation studies. In the future the developed results are expected to extend to systems with non-exponential distributed detection delays, false alarms, and non-synchronous controller.

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Chapter 4
Nonlinear Markovian Jump Systems

This chapter presents a direct robust adaptive control scheme for a class of nonlinear uncertain Markovian jump systems with nonlinear state-dependent uncertainty. In this scheme the prior knowledge of the upper bounds of the system uncertainties is not required. Furthermore, the scheme is Lyapunov-based and guarantees the closed-loop global asymptotic stability with probability one.

4.1 Introduction

Nonlinear jump systems with Markovian jumping parameters are modelled by a set of nonlinear systems with a transition between multimodels determined by a Markov chain taking values in a finite set. In [15], the problem of output feedback stabilization of a general nonlinear jump system was considered. In [14], a generic model for jump detection and identification algorithms for a class of nonlinear jump systems was proposed. The problem of disturbance attenuation with internal stability for nonlinear jump systems was discussed in [1]. Particularly, the problem of robust control for uncertain nonlinear jump systems was considered in [5], where the designed controller can guarantee the robust stability of the uncertain system, and a given disturbance attenuation can also be achieved for all admissible uncertainties. However, to the best of our knowledge, to date, in the control literature of the nonlinear uncertain jump systems even including robust stability results for the linear uncertain jump systems, an implicit assumption is that the system uncertainties can take one of the following types of uncertainties: norm bounded uncertainty, linear combination and value bounded uncertainty, and the upper bounds of those uncertainties are generally supposed to be known, and such bounds are often employed to construct some types of stabilizing state feedback controllers or some stability conditions [2–4, 6]. Actually, in the practical control problems, the bounds of the system uncertainties might not be exactly known.

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Fortunately, for such uncertain conditions on the deterministic nonlinear systems, several types of robust adaptive state feedback controller have been proposed. In [9], a robust adaptive controller is proposed to guarantee asymptotic robust stability of the system states in the face of structured uncertainty with unknown variation and structured parametric uncertainty with bounded variation. In [7], an adaptive H_∞ tracking control equipped with a VSC algorithm is proposed for a class of nonlinear multiple-input-multiple-output uncertain systems. In [17], the proposed adaptive robust continuous memoryless state feedback tracking controller with σ-modification can guarantee that the tracking error decreases asymptotically to zero. But for nonlinear uncertain Markovian jump systems, the similar results have not been reported yet in the control literature.

In this chapter, we consider a direct robust adaptive control scheme for a class of nonlinear uncertain Markovian jump systems with nonlinear state-dependent uncertainty. The proposed scheme is Lyapunov-based and guarantees the global asymptotic stability with probability one of the closed-loop systems. Compared with the existing work in the literature, our model provides a more realistic formulation which allows the switching component to depend on the continuous states by considering the x-dependent generator.

The chapter is organized as follows. In Sect. 4.2, the problem to be tackled is stated and some standard assumptions are introduced. In Sect. 4.3, a robust adaptive control scheme is proposed and the corresponding stability analysis is shown. A numerical example is presented in Sect. 4.4 to support our theoretical results. Finally this chapter will be concluded in Sect. 4.5 with a brief discussion of the results.

4.2 Description of Nonlinear Uncertain Jump System

Consider the following piecewise nonlinear uncertain jump system:

$$\begin{aligned} \dot{x}(t) &= f(x(t), r_t) + \Delta f(x(t), r_t) \\ &\quad + G(x(t), r_t)\left[G^0(x(t), r_t) + \Delta G(x(t), r_t)\right]u(t), \end{aligned} \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{m_1}$ is the control input. The parameter r_t is continuous-time Markov process on the probability space which takes values in the finite discrete state-space $\mathscr{S} = \{1, 2, \cdots, N\}$ with generator $\Pi = [\pi_{ij}(x(t))]_{N\times N} \quad (i, j \in \mathscr{S})$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \pi_{ij}(x(t))\Delta + o(\Delta), & i \neq j \\ 1 + \pi_{ii}(x(t))\Delta + o(\Delta), & i = j \end{cases} \tag{4.2}$$

where

$$\lim_{\Delta\to 0} o(\Delta)/\Delta = 0 \qquad (\Delta > 0),$$

π_{ij} is the transition rate from i to j, and

$$\pi_{ii}(x(t)) = -\sum_{j\neq i}\pi_{ij}(x(t)) \quad (\pi_{ij}(x(t)) \geq 0,\ j \neq i),$$

and r_t is assumed to be exactly known at each time.

The functions $f(\cdot, r_t)$, $\Delta f(\cdot, r_t) : \mathbb{R}^n \to \mathbb{R}^n$ with $f(0, r_t) = 0$, $G(\cdot, r_t) : \mathbb{R}^n \to \mathbb{R}^{n\times m_1}$ and $G^0(\cdot, r_t)$, $\Delta G(\cdot, r_t) : \mathbb{R}^n \to \mathbb{R}^{m_1\times m_1}$ are smooth $\mathscr{C}^\infty(\mathbb{R}^n \times \mathscr{S})$ functions of $x(t)$ for each value of $r_t \in \mathscr{S}$ such that the system (4.1) is well defined, that is, the only equilibrium point of the system (4.1) is $x(t) = 0$ for any initial state $x(t_0)$ and any admissible control $u(t)$, where $\mathscr{C}^\infty(\mathbb{R}^n \times \mathscr{S})$ denotes all functions on $\mathbb{R}^n \times \mathscr{S}$ which are infinitely continuously differentiable in $x(t)$ for each $r_t \in \mathscr{S}$.

We have the following simplified assumptions.

Assumption 4.1 The matrix-valued function $\Pi = [\pi_{ij}(x(t))]_{N\times N}$ is continuous at $x(t) = 0$, and it satisfies the linear growth condition as follows:

$$\pi_{ij}(x(t)) \leq \pi_{ij}(0) + \gamma_{ij}||x(t)||, \quad i, j \in \mathscr{S} \tag{4.3}$$

where γ_{ij} is an unknown scalar satisfying

$$0 < \gamma_{ij} < +\infty.$$

We further assume the information of the generator at $x(t) = 0$ (when the system is stable), $[\pi_{ij}(0)]_{N\times N}$, is always known.

Assumption 4.2 There exist unknown matrice $M^* \in \mathbb{R}^{m_1\times q_1}$, $E^* \in \mathbb{R}^{m_1\times q_2}$ and fixed functions $L(\cdot, r_t) : \mathbb{R}^n \to \mathbb{R}^{m_1\times m_1}$, $N(\cdot, r_t) : \mathbb{R}^n \to \mathbb{R}^{q_1}$, $H(\cdot, r_t) : \mathbb{R}^n \to \mathbb{R}^{m_1\times m_1}$, $T(\cdot, r_t) : \mathbb{R}^n \to \mathbb{R}^{q_2\times m_1}$ with $N(0, r_t) = 0$, for each $r_t \in \mathscr{S}$ such that

$$\Delta f(x(t), r_t) = G(x(t), r_t)L(x(t), r_t)M^*N(x(t), r_t), \tag{4.4}$$

$$\Delta G(x(t), r_t) = H(x(t), r_t)E^*T(x(t), r_t). \tag{4.5}$$

Furthermore, the input gain matrix $[G^0(x(t), r_t) + \Delta G(x(t), r_t)]$ is positive (or negative) definite for each $r_t \in \mathscr{S}$, i.e., there is a known bound $\lambda(r_t) > 0$ such that

$$\begin{cases} G^0(x(t), r_t) + \Delta G(x(t), r_t) > \lambda(r_t)I \\ or \\ G^0(x(t), r_t) + \Delta G(x(t), r_t) < -\lambda(r_t)I \end{cases} \tag{4.6}$$

Remark 4.1 The assumption (4.6) may restrict the structure of the system. However, it do fit for a class of controllable systems as indicated in [16] (see A8 in it).

Assumption 4.3 There exist unknown matrix $K^* \in \mathbb{R}^{m_1 \times p}$ and fixed functions $\widehat{G}(\cdot, r_t) : \mathbb{R}^n \to \mathbb{R}^{m_1 \times m_1}$, $F(\cdot, r_t) : \mathbb{R}^n \to \mathbb{R}^p$, with $F(0, r_t) = 0$, such that the following nonlinear jump system

$$\begin{aligned}\dot{x}(t) &= f(x(t), r_t) + G(x(t), r_t)\widehat{G}(x(t), r_t)K^* F(x(t), r_t) \\ &\triangleq f^*(x(t), r_t)\end{aligned} \tag{4.7}$$

is stochastically asymptotically stable (SAS).

And, there exist $V^*(x(t), r_t) \in \mathscr{C}^1(\mathbb{R}^n \times \mathscr{S}; \mathbb{R}_+)$ and continuous function $l(\cdot, r_t)$: $\mathbb{R}^n \to \mathbb{R}^{q_3}$ with $V^*(0, r_t) = 0, l(0, r_t) = 0$ such that

(i). for any $r_t = i \in \mathscr{S}$,
  a. $l_i^T(x) l_i(x) > 0, \ \ For \ any \ x \in \mathbb{R}^n, x \neq 0$
  b. $V^*(x(t), i)$ is continuous and has bounded first derivatives with respect to x(t) and t.
  c. for $\forall \ x(t) \neq 0$, $(\frac{\partial V_i^*(x(t))}{\partial x(t)})^T G_i(x(t)) G_i^T(x(t)) (\frac{\partial V_i^*(x(t))}{\partial x(t)})$ is invertible.

(ii). there exists $\beta_1(\cdot), \beta_2(\cdot) \in \mathscr{K}_\infty$ such that

$$\beta_1(||x(t)||) \le V^*(x(t), r_t) \le \beta_2(||x(t)||).$$

(iii). the following equations hold for all $i \in \mathscr{S}$

$$(\frac{\partial V_i^*(x(t))}{\partial x(t)})^T f_i^*(x(t)) + \sum_{j=1}^{N} \pi_{ij}(0) V_j^*(x(t)) + l_i^T(x(t)) l_i(x(t)) \le 0, \tag{4.8}$$

where $\mathscr{C}^1(\mathbb{R}^n \times \mathscr{S}; \mathbb{R}_+)$ denote all nonnegative functions $V(x(t), i)$ on $\mathbb{R}^n \times \mathscr{S}$ which are continuously differentiable in $x(t)$.

For the sake of simplicity, we denote the current regime by an index (e.g. $f_i(\cdot)$ stands for $f(\cdot, r_t)$ when $r_t = i \in \mathscr{S}$).

Remark 4.2 This assumption is similar to the one in Theorem 2.1 in [16] except the coupled term $\sum_{j=1}^{N} \pi_{ij} V_j^*$ which is derived from the Markovian infinitesimal generator $\mathfrak{L}$. Since we have assumed the new closed-loop system (4.7) is SAS, this assumption is reasonable for the system [10, 11, 13], and the numerical example provided later also demonstrates the rationality of this assumption.

4.3 Robust Adaptive Control for Nonlinear Uncertain Jump Systems

In recent years, Deng Hua have established stochastic versions of the Lasalle Theorem for stochastic system with state multiplicative noises [8, 12]. Following their studies, we extend the dynamic model to jump system with Markovian jumping parameter, and establish Markovian jumping versions of the well-known Lasalle stability theorem.

Theorem 4.1 *Consider the jump system*

$$\dot{x}(t) = f(x(t), u(t), r_t, t), \tag{4.9}$$

where the definitions of $x(t), r_t, u(t)$ are the same as those in system (4.1).

If there exists a Lyapunov function $V(x(t), r_t, t)$ and $\mathscr{K}_\infty$ functions $\alpha_1(\cdot), \alpha_2(\cdot)$ such that

(i). $V(0, r_t, t) = 0$.
(ii). for a fixed $r_t = i$, $V(x(t), i, t)$ is continuous and has bounded first derivatives with respect to x and t.
(iii). $\alpha_1(||x(t)||) \le V(x(t), r_t, t) \le \alpha_2(||x(t)||)$.
(iv). $\mathfrak{L}V(x(t), r_t, t) \le -W(x(t))$*, where* $W(\cdot) : \mathbb{R}^n \to \mathbb{R}$ *is continuous and nonnegative.*

Then there is a unique strong solution of (4.9) for all $x_0 \in \mathbb{R}^n (x_0 < \infty)$

$$P\{\lim_{t\to\infty} W(x(t)) = 0\} = 1. \tag{4.10}$$

Proof Since we have assumed that the system state $x(t)$ is continuous with respect to time t, then $\alpha_1(||x(t)||), \alpha_2(||x(t)||), W(x(t))$, are all continuous functions of time t. Then for any $s > 0$, we define the stopping time

$$\tau_s = inf\{t \ge 0 : ||x(t)|| \ge s\}.$$

It is easy to obtain that:

- $\tau_s \to \infty$ with probability one when $s \to \infty$
- $0 \le ||x(t)|| \le s$ when $0 \le t \le \tau_s$

Let $t_s = min\{\tau_s, t\}$ for any $t \ge 0$. The Dynkin's formula shows that

$$E[V(x(t_s), r(t_s), t_s)] \le V(x_0, r_0, 0) - E\{\int_0^{t_s} W(x(\tau))d\tau\}.$$

Consider the condition (iii) in Theorem 4.1 we have

$$E[\alpha_1(||x||)] \le \alpha_2(||x_0||) - E\{\int_0^{t_s} W(x(\tau))d\tau\}. \tag{4.11}$$

Since $\alpha_1(\cdot)$ is a $\mathscr{K}_\infty$ function, then the left of Eq. (4.11) is nonnegative. Thus

$$E\{\int_0^{t_s} W(x(\tau))d\tau\} \le \alpha_2(||x_0||).$$

Since $W(\cdot) \ge 0$, letting $s \to \infty$, $t \to \infty$, and applying Fatou's lemma yields

$$E\{\int_0^\infty W(x(\tau))d\tau\} \leq \alpha_2(||x_0||).$$

Hence the following two results hold with probability one:

$$\int_0^\infty W(x(\tau))d\tau < \infty, \tag{4.12}$$

$$\lim_{t\to\infty} V(x(t), r_t, t) \qquad \text{exists and is finite.} \tag{4.13}$$

Then there must be a probability subspace $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ exists with probability one, in which (4.12) and (4.13) always hold. Next we need to proof that in this probability subspace following limitation always holds:

$$\lim_{t\to\infty} W(x(t)) = 0. \tag{4.14}$$

Since $W(x(t))$ is a continuous function of system state $x(t)$ and time t, the rest of the proof is the same as that of Theorem 2.1 in [12] and it's omitted here.

Now, let

$$Y \triangleq \left[\, I_{m_1\times m_1},\ I_{m_1\times m_1},\ \cdots,\ I_{m_1\times m_1} \,\right]_{m_1\times q_2 m_1},$$
$$E^{**} \triangleq diag\left[\, E^{*^1},\ E^{*^2},\ \cdots,\ E^{*^{q_2}} \,\right],$$

where $E^{*^i}(t)$ denotes the ith column of E^*. Then

$$\Delta G(x(t), r_t) = H(x(t), r_t) Y E^{**} T(x(t), r_t).$$

From a practical perspective, the matrix E^{**} represents the parameters of a physical plant, so E^{**} can be assumed to be bounded, i.e., there exists a known compact set [7]

$$\Omega_2 \triangleq \{E^*(t) \mid E^{*^{i^T}}(t) E^{*^i}(t) \leq \beta, i = 1, 2, \cdots, q_2\},$$

such that

$$E^{**} \in \Omega_2.$$

Let

$$\Omega_3 \triangleq \{E^*(t) \mid E^{*^{i^T}}(t) E^{*^i}(t) \leq \beta + c, i = 1, 2, \cdots, q_2\},$$

where $\beta > 0, c > 0$ that can be arbitrarily specified by the designer.

Let $\overline{E}(t)$, $\overline{K}(t)$, $\overline{M}(t)$, $\overline{Z}(t)$ denote the estimated value of E^{**}, K^*, M^* and $Z^* = \max_{i,j\in S}\{\gamma_{i,j}\}$ respectively. Then, the smooth projection algorithm with respect to $\overline{E}(t)$ can be obtained as [16]:

$$\begin{aligned}
&Proj(\overline{E}^i(t), \Phi^i(r_t)) \\
&\triangleq \begin{cases} \Phi^i(r_t) - \frac{(\|\overline{E}^i(t)\|^2-\beta)\Phi^{i^T}(r_t)\overline{E}^i(t)}{c\|\overline{E}^i(t)\|^2}\overline{E}^i(t), \\ \qquad if \;\; \| \overline{E}^i(t) \|^2 > \beta \, and \; \Phi^{i^T}_{r_t}\overline{E}^i(t) > 0 \\ \\ \Phi^i(r_t), \qquad\qquad otherwise \end{cases} \\
&i = 1, 2, \cdots, q_2.
\end{aligned} \tag{4.15}$$

For some smooth functions $\Phi^i(r_t)$ which will be defined in Theorem 4.2. Then we have the following result.

Theorem 4.2 *Consider the nonlinear uncertain jump system (4.1) with the above assumptions, let $Q_1 \in \mathbb{R}^{p\times p}$, $Q_2 \in \mathbb{R}^{q_1\times q_1}$, $Q_3 \in \mathbb{R}^{q_2\times q_2}$ be positive definite. Then the adaptive feedback control law*

$$u(t) = \begin{cases} u_1, & if \quad G^{00} \quad \text{is invertible} \\ u_2, & if \quad G^{00} \quad \text{is not invertible} \end{cases} \tag{4.16}$$

with

$$\begin{aligned}
G^{00} &= [G^0(x(t), r_t) + H(x(t), r_t)Y\overline{E}(t)T(x(t), r_t)], \\
u_1 &= \left[G^0(x(t), r_t) + H(x(t), r_t)Y\overline{E}(t)T(x(t), r_t)\right]^{-1} \\
&\quad \times \left[\widehat{G}(x(t), r_t)\overline{K}(t)F(x(t), r_t) - L(x(t), r_t)\overline{M}(t)N(x(t), r_t)\right] \\
&\quad - \left[G^0(x(t), r_t) + H(x(t), r_t)Y\overline{E}(t)T(x(t), r_t)\right]^{-1} \times u_3,
\end{aligned} \tag{4.17}$$

$$u_2 = \begin{cases} \varepsilon(r_t)\frac{f^0(x(t),r_t)f^0(x(t),r_t)G^T(x(t),r_t)\frac{\partial V^*(x(t),r_t)}{\partial x(t)}}{\lambda(r_t)\|f^0(x(t),r_t)G^T(x(t),r_t)\frac{\partial V^*(x(t),r_t}{\partial x(t)}\|} + \frac{\varepsilon(r_t)}{\min_{r_t\in S}\{\lambda(r_t)\}}u_3, \\ \qquad if \;\; \| f^0(x(t), r_t)G^T(x(t), r_t)\frac{\partial V^*(x(t),r_t)}{\partial x(t)} \| \neq 0 \\ \frac{\varepsilon(r_t)}{\min_{r_t\in S}\{\lambda(r_t)\}}u_3, \quad if \;\; \| f^0(x(t), r_t)G^T(x(t), r_t)\frac{\partial V^*(x(t),r_t)}{\partial x(t)} \| = 0 \end{cases} \tag{4.18}$$

$$\begin{aligned}
u_3 &= \|x(t)\|\overline{Z}(t)\sum_{j=1}^{N} V_j^*(x)G_i^T(x(t))(\frac{\partial V_i^*(x(t))}{\partial x(t)}) \\
&\quad \times \left[(\frac{\partial V_i^*(x(t))}{\partial x(t)})^T G_i(x(t))G_i^T(x(t))(\frac{\partial V_i^*(x(t))}{\partial x(t)})\right]^{-1},
\end{aligned} \tag{4.19}$$

guarantees the global asymptotic stability with probability one of the systems. where

$$f^0(x(t),r_t) = \| L(x(t),r_t)\overline{M}(t)N(x(t),r_t) - \widehat{G}(x(t),r_t)\overline{K}(t)F(x(t),r_t) \|,$$

$$\varepsilon(r_t) = \begin{cases} -1, \; if \; \left[G^0(x(t),r_t) + H(x(t),r_t)YE^{**}T(x(t),r_t)\right] > 0 \\ 1, \quad if \; \left[G^0(x(t),r_t) + H(x(t),r_t)YE^{**}T(x(t),r_t)\right] < 0 \end{cases}$$

and $\overline{K}(t) \in \mathbb{R}^{m_1 \times p}$, $\overline{M}(t) \in \mathbb{R}^{m_1 \times q_1}$, $\overline{E}(t) \in \mathbb{R}^{m_1 \times q_2}$ *and* $\overline{Z}(t) \in \mathbb{R}$ *are estimated parameters with update laws:*

$$\dot{\overline{K}}(t) = -\frac{1}{2}G^{*^T}(x(t),r_t)G^T(x(t),r_t)\frac{\partial V^*(x(t),r_t)}{\partial x}F^{*^T}(x(t),r_t)Q_1^{-1}, \tag{4.20}$$

$$\dot{\overline{M}}(t) = \frac{1}{2}L^{*^T}(x(t),r_t)G^T(x(t),r_t)\frac{\partial V^*(x(t),r_t)}{\partial x}N^{*^T}(x(t),r_t)Q_2^{-1}, \tag{4.21}$$

$$\dot{\overline{Z}}(t) = \frac{1}{2}\|x(t)\|\sum_{j=1}^{N}V_j^*(x), \tag{4.22}$$

$$\dot{\overline{E}}^i(t) = \begin{cases} \frac{1}{2}Proj(\overline{E}^i(t), \Phi^i(r_t)), \\ \quad if \; [G_i^0(x) + H(x(t),r_t)Y\overline{E}(t)T(x(t),r_t)] \text{ is invertible} \\ 0, \quad otherwise \\ (i = 1, 2, \cdots, m_1, \; r_t \in \mathscr{S}) \end{cases}, \tag{4.23}$$

$$\Phi(r_t) \triangleq Y^T H^T(x(t),r_t)G^T(x(t),r_t)(\frac{\partial V^*}{\partial x})u^T T^T(x(t),r_t), \tag{4.24}$$

$\Phi^i(r_t)$, $\overline{E}^i(t)$ *denote the ith column of* $\Phi(r_t)$ *and* $\overline{E}(t)$*, respectively.*

Proof Consider the Lyapunov function candidate

$$\begin{aligned} &V(x(t),\overline{K}(t),\overline{M}(t),\overline{E}(t),\overline{Z}(t),r_t) \\ &= V^*(x(t),r_t) + tr\left[\widetilde{K}(t)Q_1\widetilde{K}^T(t)\right] + tr\left[\widetilde{M}(t)Q_2\widetilde{M}^T(t)\right] + tr\left[\widetilde{E}(t)Q_3\widetilde{E}^T(t)\right] + \widetilde{Z}^2, \end{aligned} \tag{4.25}$$

where $\widetilde{K}(t) \triangleq \overline{K}(t) - K^*$, $\widetilde{M}(t) \triangleq \overline{M}(t) - M^*$, $\widetilde{E}(t) \triangleq \overline{E}(t) - E^{**}$, $\widetilde{Z}(t) \triangleq \overline{Z}(t) - Z^*$ denote the estimated error.

Considering (4.7), we have (when $r_t = i, x(t) = x$)

$$
\begin{aligned}
\mathfrak{L}V_i(x) = &(\frac{\partial V_i^*(x)}{\partial x})^T \left\{ f_i^*(x) + G_i(x)L_i(x)M^*N_i(x) \right\} \\
&+ (\frac{\partial V_i^*(x)}{\partial x})^T G_i(x)[G_i^0(x) + H_i(x)YE^{**}T_i(x)]u \\
&- (\frac{\partial V_i^*(x)}{\partial x})^T G_i(x)\widehat{G}_i(x)K^*F_i(x) + 2tr\left[\widetilde{K}(t)Q_1\dot{\overline{K}}^T(t)\right] \\
&+ 2tr\left[\widetilde{M}(t)Q_2\dot{\overline{M}}^T(t)\right] + 2tr\left[\widetilde{E}(t)Q_3\dot{\overline{E}}^T(t)\right] + 2\dot{\overline{Z}}\widetilde{Z}(t) + \sum_{j=1}^{N}\pi_{ij}(x)V_j^*(x)
\end{aligned}
\tag{4.26}
$$

Case I. When $G_i^0(x) + H_i(x)Y\overline{E}T_i(x)$ is invertible

$$
\begin{aligned}
\mathfrak{L}V_i(x) = &(\frac{\partial V_i^*(x)}{\partial x})^T \left\{ f_i^*(x) + G_i(x)L_i(x)M^*N_i(x) \right\} \\
&+ (\frac{\partial V_i^*(x)}{\partial x})^T G_i(x)[G_i^0(x) + H_i(x)Y\overline{E}(t)T_i(x)]u \\
&- (\frac{\partial V_i^*(x)}{\partial x})^T G_i(x)H_i(x)Y\widetilde{E}(t)T_iu \\
&- (\frac{\partial V_i^*(x)}{\partial x})^T G_i(x)\widehat{G}_i(x)K^*F_i(x) + 2tr\left[\widetilde{K}(t)Q_1\dot{\overline{K}}^T(t)\right] \\
&+ 2tr\left[\widetilde{M}(t)Q_2\dot{\overline{M}}^T(t)\right] + 2tr\left[\widetilde{E}(t)Q_3\dot{\overline{E}}^T(t)\right]
\end{aligned}
\tag{4.27}
$$

By taking (4.17) and (4.20)–(4.23) into account, the above equation is equal to

$$
\begin{aligned}
\mathfrak{L}V_i(x) = &(\frac{\partial V_i^*(x)}{\partial x})^T \left[f_i^*(x) - G_i(x)L_i(x)\widetilde{M}(t)N_i(x) \right] \\
&- (\frac{\partial V_i^*(x)}{\partial x})^T G_i(x)H_i(x)Y\widetilde{E}(t)T_iu + (\frac{\partial V_i^*(x)}{\partial x})^T G_i(x)\widehat{G}_i(x)\widetilde{K}(t)F_i(x) \\
&- tr\left[\widetilde{K}(t)F_i(x)(\frac{\partial V_i^*(x)}{\partial x})^T G_i(x)\widehat{G}_i(x)\right] \\
&+ tr\left[\widetilde{M}(t)N_i(x)(\frac{\partial V_i^*(x)}{\partial x})^T G_i(x)L_i(x)\right] \\
&+ 2\dot{\overline{Z}}\widetilde{Z}(t) - \|x(t)\|\overline{Z}(t)\sum_{j=1}^{N}V_j^*(x) + \sum_{j=1}^{N}\pi_{ij}(x)V_j^*(x) \\
&+ tr\left[\widetilde{E}(t)diag^T\left[Proj(\overline{E}^j(t), \Phi_i^j)\right]\right].
\end{aligned}
\tag{4.28}
$$

From (4.15) and the proof of Lemma 1 in [16] we get

$$
\begin{aligned}
& tr\left[\widetilde{E}(t)diag^T\left[Proj(\overline{E}^j(t),\Phi_i^j)\right]\right]-(\frac{\partial V_i^*}{\partial x})^T G_i(x)H_i(x)Y\widetilde{E}(t)T_i u \\
&= tr\left[\widetilde{E}(t)diag^T\left[Proj(\overline{E}^j(t),\Phi_i^j)\right]\right]-tr\left[\widetilde{E}(t)^T Y^T H_i^T(x)G_i^T(x)\frac{\partial V_i^*}{\partial x}u^T T_i^T\right] \\
&\le 0
\end{aligned}
\tag{4.29}
$$

and $\overline{E}(t)\in\Omega_3$ if $\overline{E}(0)\in\Omega_2$.

On the other side,

$$
\begin{aligned}
\text{(i)}\quad & 2\dot{\overline{Z}}\widetilde{Z}(t)-\|x(t)\|\overline{Z}(t)\sum_{j=1}^N V_j^*(x)+\sum_{j=1}^N \pi_{ij}(x)V_j^*(x) \\
&\le \|x(t)\|\sum_{j=1}^N V_j^*(x)(\overline{Z}-Z^*)-\|x(t)\|\overline{Z}(t)\sum_{j=1}^N V_j^*(x)+\sum_{j=1}^N \pi_{ij}(0)V_j^*(x)+\sum_{j=1}^N \gamma_{ij}\|x(t)\|V_j^*(x) \\
&\le \sum_{j=1}^N \pi_{ij}(0)V_j^*(x)
\end{aligned}
\tag{4.30}
$$

$$
\begin{aligned}
\text{(ii)}\quad & -(\frac{\partial V_i^*}{\partial x})^T G_i(x)L_i(x)\widetilde{M}(t)N_i(x)+tr\left[\widetilde{M}(t)N_i(x)(\frac{\partial V_i^*}{\partial x})^T G_i(x)L_i(x)\right] \\
&= -tr\left[\widetilde{M}(t)N_i(x)(\frac{\partial V_i^*}{\partial x})^T G_i(x)L_i(x)\right]+tr\left[\widetilde{M}(t)N_i(x)(\frac{\partial V_i^*}{\partial x})^T G_i(x)L_i(x)\right]=0
\end{aligned}
\tag{4.31}
$$

$$
\begin{aligned}
\text{(iii)}\quad & (\frac{\partial V_i^*}{\partial x})^T G_i(x)\widehat{G}_i(x)\widetilde{K}(t)F_i(x)-tr\left[\widetilde{K}(t)F_i(x)(\frac{\partial V_i^*}{\partial x})^T G_i(x)\widehat{G}_i(x)\right] \\
&= tr\left[\widetilde{K}(t)F_i(x)(\frac{\partial V_i^*}{\partial x})^T G_i(x)\widehat{G}_i(x)\right]-tr\left[\widetilde{K}(t)F_i(x)(\frac{\partial V_i^*}{\partial x})^T G_i(x)\widehat{G}_i(x)\right]=0
\end{aligned}
\tag{4.32}
$$

Taking the above results into (4.28), we have

$$
\mathfrak{L}V_i \le -l_i^T(x)l_i(x). \tag{4.33}
$$

Case II. When $G_i^0(x)+H_i(x)Y\overline{E}T_i(x)$ is not invertible

$$\begin{aligned}
\mathfrak{L}V_i &= (\frac{\partial V_i^*}{\partial x})^T \left\{ f_i^*(x) + G_i(x)L_i(x)M^*N_i(x) \right\} \\
&\quad + (\frac{\partial V_i^*}{\partial x})^T G_i(x)[G_i^0(x) + H_i(x)YE^{**}T_i(x)]u \\
&\quad - (\frac{\partial V_i^*}{\partial x})^T G_i(x)\widehat{G}_i(x)K^*F_i(x) + 2tr\left[\widetilde{K}(t)Q_1\dot{\overline{K}}^T(t)\right] \\
&\quad + 2tr\left[\widetilde{M}(t)Q_2\dot{\overline{M}}^T(t)\right] + 2tr\left[\widetilde{E}(t)Q_3\dot{\overline{E}}^T(t)\right] \\
&\quad + 2\dot{\overline{Z}}\widetilde{Z}(t) + \sum_{j=1}^{N}\pi_{ij}(x)V_j^*(x) \\
&\le (\frac{\partial V_i^*}{\partial x})^T \left\{ f_i^*(x) - G_i(x)L_i(x)\widetilde{M}(t)N_i(x) \right\} \\
&\quad + (\frac{\partial V_i^*}{\partial x})^T G_i(x)[G_i^0(x) + H_i(x)YE^{**}T_i(x)]u \\
&\quad + (\frac{\partial V_i^*}{\partial x})^T G_i(x)\widehat{G}_i(x)\widetilde{K}(t)F_i(x) \\
&\quad + (\frac{\partial V_i^*}{\partial x})^T G_i(x)\left[L_i(x)\overline{M}(t)N_i(x) - \widehat{G}_i(x)\overline{K}(t)F_i(x)\right] \\
&\quad + 2tr\left[\widetilde{K}(t)Q_1\dot{\overline{K}}^T(t)\right] + 2tr\left[\widetilde{M}(t)Q_2\dot{\overline{M}}^T(t)\right] \\
&\quad + 2tr\left[\widetilde{E}(t)Q_3\dot{\overline{E}}^T(t)\right] + 2\dot{\overline{Z}}\widetilde{Z}(t) + \sum_{j=1}^{N}\pi_{ij}(x)V_j^*(x)
\end{aligned}$$

where

$$\begin{aligned}
&(\frac{\partial V_i^*}{\partial x})^T G_i(x)[G_i^0(x) + H_i(x)YE^{**}T_i(x)](u_2 - \frac{\varepsilon(r_t)}{\min\limits_{r_t\in S}\{\lambda(r_t)\}}u_3) \\
&+ (\frac{\partial V_i^*}{\partial x})^T G_i(x)\left[L_i(x)\overline{M}(t)N_i(x) - \widehat{G}_i(x)\overline{K}(t)F_i(x)\right] \\
&\le (\frac{\partial V_i^*}{\partial x})^T G_i(x)[G_i^0(x) + H_i(x)YE^{**}T_i(x)](u_2 - \frac{\varepsilon(r_t)}{\min\limits_{r_t\in S}\{\lambda(r_t)\}}u_3) \\
&+ \| f_i^0(x)G_i^T(x)\frac{\partial V_i^*}{\partial x} \| \triangleq \overline{V}.
\end{aligned}$$

From (4.18) we have:
when $\| f_i^0(x)G_i^T(x)\frac{\partial V_i^*}{\partial x} \| = 0,$

$$\overline{V} = 0;$$

when $\| f_i^0(x)G_i^T(x)\frac{\partial V_i^*}{\partial x} \| \neq 0,$

$$\overline{V} = \varepsilon_i (f_i^0(x) G_i^T(x) \frac{\partial V_i^*}{\partial x})^T [G_i^0(x) + H_i(x) Y E^{**} T_i(x)]$$
$$\times \frac{(f_i^0(x) G_i^T(x) \frac{\partial V_i^*}{\partial x})}{\lambda_i \parallel f_i^0(x) G_i^T(x) \frac{\partial V_i^*}{\partial x} \parallel} + \parallel f_i^0(x) G_i^T(x) \frac{\partial V_i^*}{\partial x} \parallel \le 0. \tag{4.34}$$

The discussion of the rest terms in Eq. (4.34) is the same as that of Case I, so we have

$$\mathfrak{L} V_i \le -l_i^T(x) l_i(x). \tag{4.35}$$

That satisfies the conditions of Theorem 4.1, so

$$P\{\lim_{t\to\infty} l_i^T(x(t)) l_i(x(t)) = 0\} = 1.$$

Since $l_i^T(x) l_i(x) > 0, \ x \in \mathbb{R}^n$, $x \neq 0$ and $l_i(0) = 0$, we have

$$P\{\lim_{t\to\infty} x(t) = 0\} = 1.$$

That is, the nonlinear uncertain jump system (4.1) is globally asymptotically stable with probability one.

Remark 4.3 Since we have assumed that $[G^0(x(t), r_t) + \Delta G(x(t), r_t)]$ is positive (or negative) definite for each $r_t \in \mathscr{S}$, robust controller u_2 can globally be used to dominate the performance of the controlled system, then the controller u_1 seems to be redundant. However, the adaptive estimation of the uncertain matrix E^*–the perturbation term of the controller gain has not been taken into account in u_2 that must affect the asymptotic stability performance of the controlled system to some degree at the beginning of the controller operation. So the application of u_1 is necessary to improve the performance of the controlled system, the numerical example provided later also demonstrates the validity of this design.

4.4 Numerical Simulation

In this section, we consider the following numerical example. A nonlinear uncertain jump system given by (4.1) in $\mathbb{R}^2$ with two regimes $r_t \in \mathscr{S} = \{1, 2\}$, where

$$
\begin{aligned}
&f_1(x(t)) = \begin{bmatrix} x_1 + 10x_2 \\ x_1 + x_1x_2 + x_1^2 \end{bmatrix}, G_1(x(t)) = \begin{bmatrix} 0 \\ \frac{1}{1+x_1^2} \end{bmatrix}, \\
&f_2(x(t)) = \begin{bmatrix} x_2 - x_1 \\ 100x_1 - 40x_2 + x_1^2 + x_2^2 \end{bmatrix}, \\
&\Delta f_1(x(t)) = \begin{bmatrix} 0 \\ \frac{1}{1+x_1^2} \end{bmatrix} x_1 \, [2 \quad 1] \begin{bmatrix} x_1 \\ x_1x_2 \end{bmatrix}, \\
&\Delta f_2(x(t)) = \begin{bmatrix} 0 \\ \frac{1}{1+x_1^2+x_1^2} \end{bmatrix} x_2 \, [2 \quad 1] \begin{bmatrix} x_2 \\ x_1x_2 \end{bmatrix}, \\
&G_2(x(t)) = \begin{bmatrix} 0 \\ \frac{1}{1+x_1^2+x_1^2} \end{bmatrix}, G_{01} = 4 + x_2^2, \\
&G_{02} = -5 - x_1^2, \quad \Delta G_1 = \cos(x_1) \begin{bmatrix} \delta_3 \; \delta_4 \end{bmatrix} \begin{bmatrix} \cos(x_1) \\ \sin(x_2) \end{bmatrix}, \\
&\Delta G_2 = \cos(x_2) \begin{bmatrix} \delta_3 \; \delta_4 \end{bmatrix} \begin{bmatrix} \cos(x_2) \\ \sin(x_1) \end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\Pi &= (\pi_{ij})(x(t)) \\
&= \begin{bmatrix} -4 - 0.5(1 + \sin(x_2))|x_1| & 4 + 0.5(1 + \sin(x_2))|x_1| \\ 2 + \frac{|x_2|}{1+x_1^2} & -2 - \frac{|x_2|}{1+x_1^2} \end{bmatrix}
\end{aligned}
$$

with

$$
\begin{aligned}
&L_1(x(t)) = x_1, \quad L_2(x(t)) = x_2, \quad M^* = [2 \quad 1], \\
&N_1(x(t)) = \begin{bmatrix} x_1 \\ x_1x_2 \end{bmatrix}, \quad N_2(x(t)) = \begin{bmatrix} x_2 \\ x_1x_2 \end{bmatrix}, \\
&H_1(x(t)) = \cos(x_1), H_2(x(t)) = \cos(x_2), E^* = \begin{bmatrix} \delta_3 \; \delta_4 \end{bmatrix}, \\
&T_1(x(t)) = \begin{bmatrix} \cos(x_1) \\ \sin(x_2) \end{bmatrix}, T_2(x(t)) = \begin{bmatrix} \cos(x_2) \\ \sin(x_1) \end{bmatrix},
\end{aligned}
$$

$\delta_1, \delta_2, \delta_3, \delta_4 \in \mathbb{R}$ are unknown with

$$
\delta_1, \delta_3 \in \begin{bmatrix} -2, \; 2 \end{bmatrix}, \delta_2, \delta_4 \in \begin{bmatrix} -1, \; 1 \end{bmatrix}.
$$

Since $G_{01}(x(t)) + \Delta G_1(x(t))$ is positive definite, and $G_{02}(x(t)) + \Delta G_2(x(t))$ is negative definite, according to (4.6), we can take $\lambda(r_t)$ as $\lambda_1 = 1, \lambda_2 = 2$.

Next, let

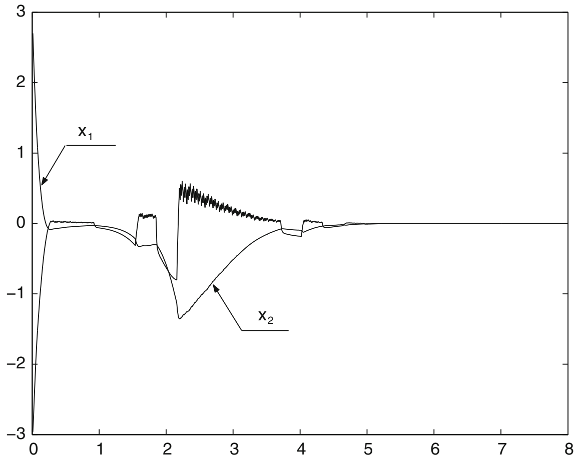

Fig. 4.1 Response of system state variable $x = [x_1 \quad x_2]^T$ with the only using of $u = u_2$

$$
\begin{aligned}
&\widehat{G}_1(x(t)) = (1 + x_1^2), \quad \widehat{G}_2(x(t)) = (1 + x_1^2 + x_2^2),\\
&K^* = \begin{bmatrix} -101 & -10 & -1 & -1 \end{bmatrix},\\
&F_1(x(t)) = \begin{bmatrix} x_1 & x_2 & x_1x_2 & x_1^2 \end{bmatrix}^T,\\
&F_2(x(t)) = \begin{bmatrix} x_1 & x_2 & x_1^2 & x_1^2 \end{bmatrix}^T,
\end{aligned} \tag{4.36}
$$

we have

$$
\begin{aligned}
f_1^*(x(t)) &= \begin{bmatrix} 1 & 10 \\ -100 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\\
f_2^*(x(t)) &= \begin{bmatrix} -1 & 1 \\ -1 & -50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\end{aligned}
$$

So that $V_1^*(x(t))$, $V_2^*(x(t))$, $l_1(x(t))$, $l_2(x(t))$ satisfying (4.8) can be given by

$$
V_1^*(x(t)) = x^T(t) \begin{bmatrix} 6.7773 & 0.6115 \\ 0.6115 & 0.6621 \end{bmatrix} x(t), \tag{4.37}
$$

$$
\begin{aligned}
V_2^*(x(t)) &= x^T(t) \begin{bmatrix} 8.2953 & 0.5664 \\ 0.5664 & 0.4356 \end{bmatrix} x(t),\\
l_1^T(x(t)) &= \begin{bmatrix} 5.6737 & 0.5848 \\ 0.5848 & 2.3744 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\\
l_2^T(x(t)) &= \begin{bmatrix} 9.7962 & 0.8674 \\ 0.8674 & 3.8343 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},
\end{aligned}
$$

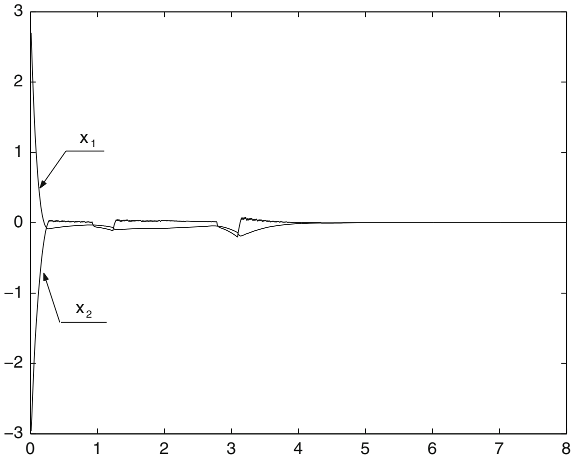

Fig. 4.2 Response of system state variable $x = [x_1 \quad x_2]^T$ with the combined using of $u = \{u_1, u_2\}$

$$x_1(0) = 2, \qquad x_2(0) = -2, \qquad \overline{K}(0) = \begin{bmatrix} -5 & -5 & 0 & 0 \end{bmatrix},$$
$$\overline{E}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \overline{M}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}, \overline{Z}(0) = 1.$$

Simulation results corresponding to the following initial conditions and design parameters are shown in Figs. 4.1 and 4.2.

It can be observed from Figs. 4.1 and 4.2 that both the adaptive robust controllers u_2 or u can indeed guarantee the asymptotic stability with probability one of the closed-loop system. On the other hand, it can be known that the combined controller $u = \{u_1, u_2\}$ which considers the adaptive estimation of the controller perturbation matrix E at the beginning of the controller operation has a rather better dynamical performance.

4.5 Summary

In this chapter, we investigated the problem of robust and adaptive control for a class of nonlinear uncertain Markovian jump systems with nonlinear state-dependent uncertainty. For such systems, a direct memoryless adaptive robust state feedback controller has been proposed. Based on the Lyapunov stability theory, it has been shown that the nonlinear uncertain closed-loop Markovian jump systems resulting from the proposed control schemes are globally asymptotically stable with probability one. However, an implicit assumption inherent in the above references is that the current regime of the jumping parameter r_t is available on-line, through a perfect observation channel, a possible direction for future work is to do the above research under the condition of less knowledge of the current regime.

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Chapter 5
Practical Stability

This chapter investigatesstochastic systems with Markovian jump parameters and time-varying delays in terms of their practical stability in probability and in the pth mean, and the practical controllability in probability and in the pth mean, respectively. Sufficient conditions are obtained by applying the comparison principle and the Lyapunov function methods. Besides, for a class of stochastic nonlinear systems with Markovian jump parameters and time-varying delays, existence conditions of optimal control are discussed. For linear systems with quadratic performance index and jumping weighted parameters, optimal control is also discussed.

5.1 Introduction

For Markovian jump systems, Lyapunov stability is now well known and has been studied widely [9, 10, 22]. Whereas, in many real world applications, the systems may be asymptotic unstable, but stay nearby a state with an acceptable fluctuation. To deal with this situation, LaSalle and Lefschetz introduce the concept of practical stability [5]. By means of examples, Lakshmikantham demonstrated that practical stability is more suitable and desirable in practice [4]. Compared with the classical Lyapunov stability theory, practical stability can depict not only the qualitative behavior but also the quantitative property, such as specific trajectory bounds and specific transient behavior. Thus, it has been widely studied in both deterministic and stochastic framework [5, 15, 16]. However, for stochastic nonlinear systems with both jump parameters and time-delays, no much progress has been seen on practical stability or practical stabilization.

For systems with Markovian jump parameters, the jump linear quadratic optimal control problem has been considered in [3, 12] by using state feedback and output feedback, respectively. A detailed discuss on optimal control of linear Markovian jump systems was given in [11]. For systems with time-delays, there are also many

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works on optimal control, including [2, 8]. However, for systems with both jump parameters and time-delays, no much progress has been seen on optimal control. This may be due to the coupling effect of the Markovian jump parameters and the time-delays, which bring essential difficulty into the analysis.

In this chapter, we will focus on a class of general stochastic systems, which are with not only jump parameters but also time-varying delays. For such class of systems, the concepts and criteria of practical stability in the pth mean and in probability, and practical controllability in probability and in the pth mean, are given. In addition, optimal control for a class of stochastic nonlinear systems with both jump parameters and time-delays is studied and some sufficient conditions for the existence of optimal control are given. Particularly, for linear systems, optimal control and the corresponding index value are provided for a class of quadratic performance indices with jumping weighted parameters.

The remainder of this chapter is organized as following: Sect. 5.2 provides some notations and preliminary results. Section 5.3 gives the comparison principle. Section 5.4 introduces the notations of practical stability in probability and in the pth mean, and presents the corresponding criteria. Section 5.5 introduces the concepts of practical controllability in probability and in the pth mean, and gives the corresponding criteria. Section 5.6 focuses on the optimal control problem. Section 5.7 includes some concluding remarks.

5.2 Markovian Jump Nonlinear Systems with Time Delays

Consider the following n-dimensional stochastic nonlinear systems with both Markovian jump parameters and time-delays:

$$\begin{aligned} dx(t) &= f(x(t), x(t-\tau(t)), t, r(t))dt \\ &\quad + g(x(t), x(t-\tau(t)), t, r(t))dB(t)\ ,\ t \geq 0, \end{aligned} \tag{5.1}$$

where initial data $\{x(\theta) : -2\mu \leq \theta \leq 0\} = \xi \in \mathscr{C}^b_{\mathscr{F}_0}([-2\mu, 0]; \mathbb{R}^n)$, $\tau(t) : \mathbb{R}_+ \to [0, \mu]$ is a Borel measurable function; $r(t)$ is a continuous-time discrete-state Markov process taking values in a finite set $\mathscr{S} = \{1, 2, \ldots, N\}$ with transition probability matrix $P = \{p_{ij}\}$ given by

$$\begin{aligned} p_{ij}(\Delta) &= P\{r(t+\Delta) = j \mid r(t) = i\} \\ &= \begin{cases} \pi_{ij}\Delta + o(\Delta), & i \neq j; \\ 1 + \pi_{ii}\Delta + o(\Delta), & i = j, \end{cases} \quad \Delta > 0. \end{aligned}$$

Here $\pi_{ij} \geq 0$ is the transition rate from i to j $(i \neq j)$, and $\pi_{ii} = -\sum_{j=1, j\neq i}^{N} \pi_{ij}$. For any given $i \in \mathscr{S}$, $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathscr{S} \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathscr{S} \to \mathbb{R}^{n\times r}$ are smooth enough to guarantee the system exist a unique solution $x(\theta, t_0, \xi)$, which satisfies $E(\sup_{t_0-\mu\leq\theta\leq t} |x(\theta, t_0, \xi)|^l) < \infty$, $\forall\ t \geq t_0$, $l \geq 0$ [10];

$B(t)$ is an r-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$, with Ω being a sample space, $\mathscr{F}$ being a σ-field, $\{\mathscr{F}_t\}_{t\geq 0}$ being a filtration and P being a probability measure.

For any given $V \in \mathscr{C}^{2,1}(\mathbb{R}^n \times [-\mu, +\infty) \times \mathscr{S}, \mathbb{R}_+)$, define an operator $\mathfrak{L}V$ from $\mathbb{R}^n \times [-\mu, +\infty) \times \mathscr{S}$ to $\mathbb{R}$ by

$$\mathfrak{L}V(x,t,i) = \frac{\partial V(x,t,i)}{\partial t} + \frac{\partial V(x,t,i)}{\partial x} f(x,y,t,i) + \frac{1}{2} tr[g^T(x,y,t,i)\frac{\partial^2 V(x,t,i)}{\partial x^2} g(x,y,t,i)] + \sum_{j=1}^{N} \pi_{ij} V(x,t,j),$$

5.3 Comparison Principle

Consider the following equation

$$\dot{\sigma}(t) = h(t, \sigma(t), \sigma_t), \quad \sigma_{t_0} = \psi, \tag{5.2}$$

where $\sigma_t = \sigma_t(\theta) = \sigma(t+\theta),\ \theta \in [-\mu, 0];\ \psi \in \mathscr{C}([-\mu, 0], \mathbb{R}_+);\ h : \mathbb{R} \times \mathbb{R}_+ \times \mathscr{C}([-\mu, 0], \mathbb{R}_+) \to \mathbb{R}$ is a continuous mapping, $h(t, \sigma, v)$ is nondecreasing with respect to v for fixed $(t, \sigma) \in \mathbb{R} \times \mathbb{R}_+$, and $h(t, 0, 0) = 0$. Denote by $\sigma_t(t_0, \psi) = \sigma(t+\theta, t_0, \psi), \theta \in [-\mu, 0],\ t \geq t_0$, the solutions of (5.2) with an initial data $\sigma_{t_0} = \psi$. Denote by $\bar{\sigma}(t, t_0, \psi)$ the largest solution [21] of (5.2) with $\sigma_{t_0} = \psi$.

Lemma 5.1 [6] *Assume that there exists a* $V \in \mathscr{C}^{2,1}(\mathbb{R}^n \times [-\mu, +\infty) \times \mathscr{S}, \mathbb{R}_+)$ *such that for the function* h *in (5.2) and any solution* $x(t) = x(t, t_0, \xi)$ *of (5.1),* $E\{V(x(t), t, i)\}$ *exists for* $t \geq t_0 - \mu$, *and*

(A₁) $\mathfrak{L}V(x,t,i) \leq h(t, V(x,t,i), V_t)$, *where* $V_t = V(x(t+\theta), t+\theta, r(t+\theta))$, $\theta \in [-\mu, 0],\ i \in \mathscr{S}$.

(A₂) $E\{h(t, V(x,t,i), V_t)\} \leq h(t, E\{V(x,t,i)\}, E\{V_t\}), t \in \mathbb{R}$.

If $E\{V(x(t_0+s), t_0+s, r(t_0+s))\} \leq \psi(s),\ s \in [-\mu, 0],\ r(t_0+s) \in \mathscr{S}$, *then* $E\{V(x(t), t, i)\} \leq \bar{\sigma}(t, t_0, \psi),\ t \geq t_0 - \mu$.

Remark 5.1 It is well known that, using the Lyapunov function method, one can get the property of the solution without solving the equation. Here, based on the condition (**A₁**) on the function V, we first construct the comparison system (5.2) which is a time-delayed nonlinear system without stochastic characteristic. Then, we can get the properties of V and the solutions of stochastic nonlinear systems (5.1). In other words, one can easily get the properties of the solutions of a complicated system (5.1) by combining Lyapunov function method and comparison principle.

5.4 Practical Stability

For convenience, we shall introduce the following definitions:

Definition 5.1 System (5.1) is said to be practically stable in probability (PSiP), if for any given $\delta > 0$, there is a pair of positive numbers (λ, ρ), $0 < \lambda < \rho$, such that for some $t_0 \in \mathbb{R}_+$ and any initial data ξ satisfying $E\{\|\xi\|\} < \lambda$, $P\{|x(t, t_0, \xi)| \geq \rho\} < \delta$, $\forall\, t \geq t_0 - \mu$.

System (5.1) is said to be uniformly practically stable in probability (UPSiP), if the system is PSiP for all $t_0 \in \mathbb{R}_+$ uniformly.

Definition 5.2 Let positive number pair (λ, ρ), $0 < \lambda < \rho$, and t_0 be given. Then the system (5.1) is said to be practically stable in the pth mean (PSpM) with respect to (λ, ρ, t_0), if for any given initial data ξ satisfying $E\{\|\xi\|^p\} < \lambda$, $E\{|x(t, t_0, \xi)|^p\} < \rho$, $\forall\, t \geq t_0 - \mu$.

System (5.1) is said to be uniformly practically stable in the pth mean (UPSpM) with respect to (λ, ρ), if the system is PSpM for all $t_0 \in \mathbb{R}_+$ uniformly.

As for the notions of practical stability for deterministic system (5.2), we can refer to [4, 16].

Definition 5.3 A function $\varphi(u)$ is said to belong to the class $\mathscr{K}$ if $\varphi \in \mathscr{C}(\mathbb{R}_+, \mathbb{R}_+)$, $\varphi(0) = 0$ and $\varphi(u)$ is strictly increasing in u. A function $\varphi(u)$ is said to belong to the class $\mathscr{V}\mathscr{K}$ if φ belongs to $\mathscr{K}$ and φ is convex. A function $\varphi(t, u)$ is said to belong to the class $\mathscr{C}\mathscr{K}$ if $\varphi \in \mathscr{C}(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $\varphi(t, 0) = 0$ and $\varphi(t, u)$ is concave and strictly increasing in u for each $t \in \mathbb{R}^+$.

The following theorems are on the criteria of practical stability.

Theorem 5.1 *Under the notations of Lemma 5.1, suppose that* $(\mathbf{A_1})$ *and* $(\mathbf{A_2})$ *hold, and there exist a function* $b \in \mathscr{K}$ *and a function* $a \in \mathscr{C}\mathscr{K}$ *such that*

$$b(|x(t)|) \leq V(x(t), t, i) \leq a(t, \|x_t\|), \ \forall\, i \in \mathscr{S}. \tag{5.3}$$

If system (5.2) is practically stable with respect to $(\lambda_1, b(\rho_1), t_0)$*, then system (5.1) is* PSiP.

Proof By (5.3) we have

$$\begin{aligned} 0 \leq E\{b(|x(t+\theta)|)\} &\leq E\{V(x(t+\theta), t+\theta, i)\} \\ &\leq E\{a(t+\theta, \|x_{t+\theta}\|)\} \\ &\leq a(t+\theta, E\{\|x_{t+\theta}\|\}). \end{aligned}$$

Here Jensen inequality has been used to get the last inequality. Because the condition that comparison system (5.2) is practically stable with respect to $(\lambda_1, b(\rho_1), t_0)$, we have that for any initial data ψ satisfying $\|\psi\| < \lambda_1$, $|\sigma(t, t_0, \psi)| < b(\rho_1)$, $\forall\, t \geq t_0 - \mu$. Therefore, $\bar{\sigma}(t, t_0, \psi) < b(\rho_1)$, $t \geq t_0 - \mu$.

Noticing that for any $\delta \in (0, 1)$, there always exists $\rho = \rho(\delta)$ such that $b(\rho_1) \le \delta b(\rho)$, then $\bar{\sigma}(t, t_0, \psi) < \delta b(\rho),\ \forall\, t \ge t_0 - \mu$. Choose $\psi = a(t_0 + \theta, E\{\|x_{t_0+\theta}\|\})$, $\theta \in [-\mu, 0]$. As $a \in \mathscr{CK}$, there exists a λ such that for any $x_{t_0+\theta}$ satisfying $E\{\|x_{t_0+\theta}\|\} < \lambda$, $0 < \|\psi\| < \lambda_1$. This together with Lemma 5.1 gives

$$E\{V(x(t), t, i)\} \le \bar{\sigma}(t, t_0, \psi) < \delta b(\rho),\ \forall\, t \ge t_0 - \mu. \tag{5.4}$$

By Tchebycheff inequality and (5.4), we can obtain

$$P\{V(x(t), t, i) \ge b(\rho)\} \le E\{V(x(t), t, i)\}/b(\rho) < \delta,\ \forall\, t \ge t_0 - \mu,$$

which together with (5.3) leads to

$$\begin{aligned} P\{|x(t, t_0, \xi)| \ge \rho\} &= P\{b(|x(t, t_0, \xi)|) \ge b(\rho)\} \\ &\le P\{V(x(t), t, i) \ge b(\rho)\} \\ &< \delta,\ \forall\, t \ge t_0 - \mu. \end{aligned}$$

Hence, system (5.1) is PSiP.

Theorem 5.2 *Under the conditions of Theorem 5.1, if $a(t, x) = a(x)$ and equation* (5.2) *is uniformly practically stable with respect to* $(\lambda_1, b(\rho))$*, then system (5.1) is* UPSiP.

Proof From the proof of Theorem 5.1, when $a(t, x) = a(x)$, λ is independent of t_0. Thus, system (5.1) is UPSiP.

Theorem 5.3 *Under the notations of Lemma 5.1, suppose that* $(\mathbf{A_1})$ *and* $(\mathbf{A_2})$ *hold, and there exist a function $b \in \mathscr{VK}$ and a function $a \in \mathscr{CK}$ such that*

$$b(|x(t)|^p) \le V(x(t), t, i) \le a(t, \|x_t\|^p),\ \forall\, i \in \mathscr{S}, \tag{5.5}$$

and for given t_0 and (λ, ρ), $0 < \lambda < \rho$, $a(t_0 + s, \lambda) < b(\rho)$, $\forall\, s \in [-\mu, 0]$. *If equation (5.2) is practically stable with respect to* $(\alpha, b(\rho), t_0)$*, then system (5.1) is* PSpM *with respect to* (λ, ρ, t_0)*, where* $\alpha = \sup_{s\in[-\mu,0]} a(t_0 + s, \lambda)$.

Proof By (5.5) and Jensen inequality, for $\forall\, \theta \in [-\mu, 0]$ and $\forall\, t \ge t_0$, we have

$$\begin{aligned} 0 \le E\{b(|x(t+\theta)|^p)\} &\le E\{V(x(t+\theta), t+\theta, i)\} \\ &\le E\{a(t+\theta, \|x_{t+\theta}\|^p)\} \\ &\le a(t+\theta, E\{\|x_{t+\theta}\|^p\}), \end{aligned}$$

which implies that $E\{V(x(t), t, i)\}$ exists for all $t \ge t_0 - \mu$. By Lemma 5.1, when $E\{V(x(t_0 + s), t_0 + s, r(t_0 + s))\} \le \psi(s)$, $\forall\, s \in [-\mu, 0]$, we have

$$E\{V(x(t), t, i)\} \le \bar{\sigma}(t, t_0, \psi), \quad \forall\, t \ge t_0 - \mu. \tag{5.6}$$

Suppose that system (5.2) is practically stable with respect to $(\alpha, b(\rho), t_0)$. Then, for $(\alpha, b(\rho))$, $\|\psi\| < \alpha$ implies $\bar{\sigma}(t, t_0, \psi) < b(\rho)$, $\forall\, t \geq t_0$. Now we claim that system (5.1) is PSpM with respect to (λ, ρ, t_0), i.e., if $E\{\|\phi\|^p\} < \lambda$, then $E\{|x(t, t_0, \phi)|^p\} < \rho$, since, otherwise, there would exist $t_1 > t_0$ and a solution $x(t, t_0, \phi)$ of system (5.1) which satisfies that $E\{\|\phi\|^p\} < \lambda$ and $E\{|x(t_1, t_0, \phi)|^p\} = \rho$. Choose $\psi(s) = a(t_0 + s, E\{\|\phi_s\|^p\})$, $\forall\, s \in [-\mu, 0]$. Then, by (5.5) we would have

$$E\{V(x(t_0 + s), t_0 + s, r(t_0 + s))\} \leq \psi(s) < \alpha, \quad s \in [-\mu, 0], \quad \|\psi\| < \alpha.$$

Consequently,

$$E\{V(x(t), t, i)\} \leq \bar{\sigma}(t, t_0, \psi) < b(\rho), \ \forall\, t \geq t_0.$$

This results in the following contradictory

$$b(\rho) = b(E\{|x(t_1, t_0, \phi)|^p\}) \leq E\{V(x(t_1), t_1, r(t_1))\} < b(\rho).$$

Thus, system (5.1) is PSpM with respect to (λ, ρ, t_0).

Theorem 5.4 *Under the conditions of Theorem 5.3, if $a(t, x) = a(x)$ and equation (5.2) is uniformly practically stable with respect to $(\alpha, b(\rho))$, then system (5.1) is* UPSpM *with respect to (λ, ρ).*

Proof From the proofs of Theorems 5.2 and 5.3, the result can be proved straightforward. Thus, the details are omitted here.

Remark 5.2 Different from the Lyapunov stability which focuses on the qualitative behavior of systems, practical stability focuses on the quantitative properties, and so is the PSiP except that the preassigned positive numbers (λ, ρ) are dependent on the size of probability δ. In addition, both practical stability and PSiP do not require that the systems have equilibria.

To illustrate the validity of Theorem 5.4, we give the following simple numerical example.

Example 5.1 Let us consider a Markovian jump linear stochastic systems

$$dx = A(t, r(t))xdt + B(t, r(t))xdB, \tag{5.7}$$

with the following specifications: $r(t)$ is a continuous-time discrete-state Markov process taking values in $\mathscr{S} = \{1, 2\}$ with transition rate matrix $\Pi = \{\pi_{ij}\}$ given by $\Pi = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$, and $A(t, 1) = -1 + \gamma_1(t)$, $A(t, 2) = -2 + \gamma_2(t)$, $B(t, 1) = B(t, 2) \equiv 1$, where γ_1 and γ_2 are real-valued functions representing parameter disturbances.

Taking Lyapunov function $V(x) = x^2$ and applying infinitesimal generator along with system (5.7), we have

$$\begin{cases} \mathfrak{L}V(x) = -x^2 + 2\gamma_1(t)x^2, & \text{for } r(t) = 1; \\ \mathfrak{L}V(x) = -3x^2 + 2\gamma_2(t)x^2, & \text{for } r(t) = 2, \end{cases}$$

which implies $\mathfrak{L}V(x) \le -V(x) + |\bar{\gamma}(t)|V(x)$, where $|\bar{\gamma}(t)| = \max\{|\gamma(t)|_1, |\gamma(t)|_2\}$.

Applying Theorem 5.4, we have the following conclusion:

Let λ and ρ $(0 < \lambda < \rho)$ be given. If $|\bar{\gamma}(t)|$ satisfies

$$\int_{t_0}^{t} (-1 + |\bar{\gamma}(\tau)|)d\tau < \ln(\frac{\rho^2}{\lambda^2}), \ \forall\, t \ge t_0, \tag{5.8}$$

then system (5.7) is UPSpM (p=2) with respect to (λ, ρ).

Remark 5.3 From Example 5.1, we can find that: (i), due to a common Lyapunov function taken, the transition rate matrix has no effect on the conclusion; (ii), the inequality (5.8) can also be written as

$$\int_{t_0}^{t} |\bar{\gamma}(\tau)|d\tau < \ln(\frac{\rho^2}{\lambda^2}) + (t - t_0), \ \forall\, t \ge t_0, \tag{5.9}$$

from which, how the practical stability boundary affects the upper bound of the disturbance can be seen; (iii), the UPSpM (p = 2) of system (5.7) can be guaranteed by (5.8) or (5.9), there is no need for the assumption of sign definiteness on the infinitesimal generator of the Lyapunov function.

5.5 Practical Controllability

In this section, we will consider the practical controllability of a class of stochastic nonlinear systems with jump parameters and time-delays.

Suppose the system is of the following form

$$\begin{aligned} dx(t) = {} & f(x(t), x(t - \tau(t)), t, r(t), u(t))dt \\ & + g(x(t), x(t - \tau(t)), t, r(t))dB(t)\ , \ t \ge t_0, \end{aligned} \tag{5.10}$$

where $u(t)$ is input, and is supposed to guarantee the existence and uniqueness of the solution process.

For convenience, we introduce the following definitions:

Definition 5.4 System (5.10) is said to be practically controllable in probability γ (PCiP-γ) with respect to (λ, β) if there exist finite time T and a control $u(\cdot)$ defined on $[t_0, T]$ such that all the solutions $x(t) = x(t, t_0, \xi, r_0, u)$ that exit from $\{x \in \mathbb{R}^n : \|\xi\| < \lambda\}$ enter into bounded region $\{x \in \mathbb{R}^n : \|x_t\| < \beta\}$ at time T instant with probability no less than $1 - \gamma$.

Definition 5.5 System (5.10) is said to be practically controllable in the pth mean (PCpM) with respect to (λ, β) if there exist a finite time T and a control $u(\cdot)$ defined on $[t_0, T]$ such that all the solutions $x(t) = x(t, t_0, \xi, r_0, u)$ that exit from $\{x \in \mathbb{R}^n : E\{\|\xi\|^p\} < \lambda\}$ enter into the bounded region $\{x \in \mathbb{R}^n : E\{\|x_t\|^p\} < \beta\}$ at the time T instant, i.e., $E\{\|\xi\|^p\} < \lambda$ implies $E\{\|x_T\|^p\} < \beta$.

The following theorems are on the criteria of practical controllability.

Theorem 5.5 *Assume that there exists a control law u for system (5.10) such that the conditions of Theorem 5.1 are satisfied, and there exists a $T = T(t_0, \psi)$ such that*

$$\bar{\sigma}(T + s, t_0, \psi) < \gamma b(\beta), \ \forall s \in [-\tau(T), 0], \tag{5.11}$$

where $\bar{\sigma}(t, t_0, \psi)$ is the maximum solution of system (5.2) with initial data (t_0, ψ), $\beta \in (0, \rho)$ is a preassigned constant. Then, system (5.10) is PCiP-γ *with respect to (λ, β).*

Proof By (5.11) and (5.4) we have

$$E\{V(x(T+s), T+s, r(T+s))\} \leq \bar{\sigma}(T+s, t_0, \psi) < \gamma b(\beta).$$

Then, by Tchebycheff inequality, we have

$$\begin{aligned} &P\{V(x(T+s), T+s, r(T+s)) \geq b(\beta)\} \\ &\leq E\{V(x(T+s), T+s, r(T+s))\}/b(\rho) < \gamma, \end{aligned}$$

which together with (5.3) leads to

$$\begin{aligned} &P\{|x(T+s, t_0, \xi)| \geq \beta\} \\ &= P\{b(|x(T+s, t_0, \xi)|) \geq b(\beta)\} \\ &\leq P\{V(x(T+s), T+s, r(T+s)) \geq b(\beta)\} < \gamma. \end{aligned}$$

Thus, system (5.10) is PCiP-γ with respect to (λ, β).

Theorem 5.6 *Assume that there exists a control law u for system (5.10) such that the conditions of Theorem 5.3 are satisfied, and there exists a $T = T(t_0, \psi)$ such that*

$$\bar{\sigma}(T+s, t_0, \psi) < b(\beta),\ \forall\, s \in [-\tau(T), 0], \tag{5.12}$$

where $\bar{\sigma}(t, t_0, \psi)$ is the maximum solution of system (5.2) with initial data (t_0, ψ), $\beta \in (0, \rho)$ is a preassigned constant. Then, system (5.10) is PCpM *with respect to (λ, β).*

Proof By (5.12) and (5.6) we have

$$\begin{aligned} b(E\{|x(T+s)|^p\}) &\leq E\{V(x(T+s), T+s, r(T+s))\} \\ &\leq \bar{\sigma}(T+s, t_0, \psi) \\ &< b(\beta). \end{aligned}$$

Thus, $E\{|x(T+s)|^p\} < \beta$.

5.6 Optimal Control

This section focuses on the optimal stabilization of n-dimensional stochastic nonlinear systems with jump parameters and time-delays. Precisely, we will consider system (5.10) and seek for a control law u to minimize the following performance index

$$\begin{aligned} J_{t_0,\xi,r_0}(u) =& E\left\{\int_{t_0}^{\infty} G(t, V(x(t, t_0, \xi, r_0, u), t, r(t)), x(t, t_0, \xi, r_0, u), r(t),\right. \\ & \left. u(t, x(t, t_0, \xi, r_0, u))))dt \middle| t_0, \xi, r_0 \right\}, \end{aligned} \tag{5.13}$$

where the function G satisfies

$$\dot{v} = -G(t, v, E\{x(t)\}, r(t), E\{u(t)\}), \quad v(t_0) = v_0 \geq 0,\ r(t_0) = r_0 \tag{5.14}$$

and $G \in \mathscr{C}[\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathscr{S} \times \mathbb{R}^m, \mathbb{R}_+]$, $G(t, 0, E\{x(t)\}, r(t),\ E\{u(t)\}) \equiv 0$, is concave in v, $E\{x(t)\}$ and $E\{u(t)\}$, and nondecreasing in v for fixed $(t, E\{x(t)\}, r(t), E\{u(t)\}) \in [t_0, \infty) \times \mathbb{R}^n \times \mathscr{S} \times \mathbb{R}^m$, nondecreasing in $E\{x(t)\}$, $x(t) = x(t, t_0, \xi, r_0, u)$. $\bar{v}(t, t_0, v_0, E\{x(t_0)\}, r_0)$ denotes the maximum solution of system (5.14) with initial data $v_0,\ E\{x(t_0)\},\ r_0$.

To this end, we now introduce the set $\mathscr{U}$ of admissible controls.

Definition 5.6 By admissible control set $\mathscr{U}$ we mean the set consisting of such control $u(t, x_t)$ that has the following properties:

(i) $u(t, x_t)$ is adapted to the σ-algebra generated by $\{x_t, r(t), t \geq t_0\}$ and $u(t, 0) = 0$;

(ii) for any given initial value $\xi \in \mathscr{C}^b_{\mathscr{F}_0}([-2\mu, 0]; \mathbb{R}^n)$ and $r_0 \in \mathscr{S}$, under $u(t, x_t)$ the system (5.10) has a unique solution $x(t) = x(t, t_0, \xi, r_0, u)$ and $E\{|x(t)|^p\} \to 0$, $t \to \infty$.

Theorem 5.7 *Suppose that (5.5) holds. If*

(i) $\mathfrak{L}V(x(t), t, i) + G(t, V(x(t), t, i), x(t), i, u) \geq 0$, $\forall\, i \in \mathscr{S}$, $\forall\, t \geq t_0$, $\forall\, u \in \mathscr{U}$*, and, moreover, there exists a* $u^0 = u^0(t, x_t)$ *such that*

(ii) $\mathfrak{L}V(x(t), t, i) + G(t, V(x(t), t, i), x(t), i, u^0) \equiv 0$, $\forall\, i \in \mathscr{S}$, $\forall\, t \geq t_0$*;*

(iii) $dx(t) = f(x(t), x(t - \tau(t)), t,$ $r(t), u^0(t))dt + g(x(t), x(t - \tau(t)), t, r(t))dB(t)$*, with* $x_{t_0} = \xi$ *and* $r(t_0) = r_0$*, has a unique solution* $x^0(t)$, $t \geq t_0$*;*

(iv) $\dot{v} = -G(t, v, E\{x^0(t)\}, r(t), E\{u^0(t)\})$ *with* $v(t_0) = v^0 \geq 0$ *is practically stable with respect to* $(\alpha, b(\rho))$*, and has a maximum solution* $\bar{v}(t) = \bar{v}(t, t_0, v_0, x_0, r_0)$ *on* $[t_0, \infty)$ *satisfying* $\lim_{t \to \infty} \bar{v}(t, t_0, v_0, x_0, r_0) = 0$*, where* $(\alpha, b(\rho))$ *is given in Theorem 5.3,then (1)* $u^0 \in \mathscr{U}$*; (2)* $J_{t_0,\xi,r_0}(u^0) = \min_{u \in \mathscr{U}} J_{t_0,\xi,r_0}(u) = E\{V(x_0, t_0, r_0)\}$*, and (3) under* u^0*, system (5.10) is* PSpM *with respect to* (λ, ρ).

Proof Let $x^0(t) = x(t, t_0, \xi, r_0, u^0)$ denote the solution of system (5.10) under the control u^0 satisfying the condition (ii). Then, by the proof of Theorem 5.3, we know that system (5.10) is PSpM with respect to (λ, ρ), and

$$E\{V(x^0(t), t, i)\} \leq \bar{v}(t, t_0, v_0, x_0, r_0) \to 0. \tag{5.15}$$

Further, by (5.5) we have $b(E\{|x^0(t)|^p\}) \to 0$, $E\{|x^0(t)|^p\} \to 0$, $t \to \infty$. Therefore, (1) and (3) hold.

For any given admissible control $u \in \mathscr{U}$, let $x(t) = x(t, t_0, \xi, r_0, u)$ be the corresponding solution of system (5.10). Then, by

$$E\{V(x(t), t, r(t))\} - E\{V(x(t_0), t_0, r_0)\} = E\{\int_{t_0}^{t} \mathfrak{L}V(x(s), s, r(s))ds\}$$

and condition (ii) we have

$$\begin{aligned} &E\{V(x^0(t), t, r(t))\} - E\{V(x(t_0), t_0, r_0)\} \\ &= E\{\int_{t_0}^{t} -G(s, V(x^0(s), s, r(s)), x^0(s), r(s), u^0(s))ds \Big| t_0, \xi, r_0\}. \end{aligned}$$

Letting $t \to \infty$, by (5.15) we have $J_{t_0,\xi,r_0}(u^0) = E\{V(x(t_0), t_0, r_0)\}$.

Take arbitrarily a control $u^* \in \mathscr{U}$. Then, by (5.5) and condition (ii) of Definition 5.6 we have $E\{V(x^*(t), t, r(t))\} \to 0,\ \ t \to \infty$. This together with condition (i) gives $J_{t_0,\xi,r_0}(u^*) \geq E\{V(x(t_0), t_0, r_0)\}$. Thus, by the arbitrariness of $u^* \in \mathscr{U}$ we can arrive at 2).

Remark 5.4 From Theorem 5.7, we can get Hamilton–Jacobi–Bellman equation

$$\min_{u \in \mathscr{U}} [\mathfrak{L}V(x, t, k) + G(t, V(x, t, k), x, k, u)] = 0, \tag{5.16}$$

which is similar to the result in [14]. Whereas, the results we have got are valid for more general nonlinear stochastic systems, especially, for those with both Markovian jump parameters and time-delays.

Remark 5.5 In Theorem 5.7, condition (i) and (ii) guarantee the existence of optimal control; condition (iii) guarantees the existence and uniqueness of the solutions to system (5.10) under the optimal control u^0; condition (iv) guarantees that, for system (5.10) under the optimal control u^0, the optimal index value $J_{t_0,x_0,r_0}(u^0) = E\{V(x_0, t_0, r_0)\}$, further, by the inequality (5.5), the system (5.10) is PSpM with respect to (λ, ρ).

The following example and corollary demonstrate the validity of our results.

Example 5.2 Consider the following stochastic nonlinear system

$$\begin{aligned} dx(t) = [&f(x(t), x(t - \tau(t)), t, r(t)) \\ &+ B(x(t), x(t - \tau(t)), t, r(t))u(t)]dt \\ &+ g(x(t), x(t - \tau(t)), t, r(t))dB(t), \end{aligned}$$

with initial data $\{x(\theta) : -2\mu \leq \theta \leq 0\} = x_0 \in \mathscr{C}^b_{\mathscr{F}_0}([-2\mu, 0]; \mathbb{R}^n)$, where $x \in \mathbb{R}^n$, $\tau(t) : \mathbb{R}_+ \to [0, \mu]$ is a Borel measurable function; $f \in \mathscr{C}[\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathscr{S}, \mathbb{R}^n]$, $B : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathscr{S} \to \mathbb{R}^{n \times m}$ are continuous $n \times m$ matrices; $u \in \mathbb{R}^m (m \leq n)$, $r(t) \in \mathscr{S} = \{1, 2, \ldots, N\}$ is a Markov chain.

Suppose that the corresponding control-free system

$$\begin{aligned} dx(t) = &f(x(t), x(t - \tau(t)), t, r(t))dt \\ &+ g(x(t), x(t - \tau(t)), t, r(t))dB(t), \end{aligned}$$

is practically stable. Then, there exist $V(x, t, i) \in \mathscr{C}^{2,1}(\mathbb{R}^n \times \mathbb{R} \times \mathscr{S}, \mathbb{R}^+)$, which satisfies

$$\mathfrak{L}V(x, t, i) = W(x, y, t, i) + \frac{\partial V(x, t, i)}{\partial x} B(x, y, t, i)u,$$

where

$$W(x, y, t, i) = \frac{\partial V(x,t,i)}{\partial t} + \frac{\partial V(x,t,i)}{\partial x} f(x, y, t, i)$$
$$+ \frac{1}{2} g^T(x, y, t, i) \frac{\partial^2 V}{\partial x^2} g(x, y, t, i) + \sum_{j=1}^{N} \pi_{ij} V(x, t, j) \leq 0.$$

Similar to the proof of Example 1 in [14], one can get that the control law minimizing the optimal performance index

$$J_{t_0,x_0,r_0}(u) = E\left\{ \int_{t_0}^{\infty} [Q_1(x, y, t, r(t)) + u^T Q_2(t, r(t))u \Big| t_0, x_0, r_0] dt \right\}$$

is

$$u^0 = -\frac{1}{2} Q_2^{-1}(t, r(t)) B^T(t) \frac{\partial V(x, t, r(t))}{\partial x},$$

where

$$Q_1(x, y, t, r(t)) = -W(x, y, t, r(t)) + [u^0]^T Q_2(t, r(t))u^0,$$

the optimal index value is $J_{t_0,x_0,r_0} = E\{V(x(t_0), t_0, r_0)\}$.

Example 5.3 We now solve the optimal control problem for a class of linear systems with jump parameters and time-delays. Suppose the system is of the form

$$\dot{x}(t) = A(t, r(t))x(t) + B(t, r(t))x(t-h) + D(t, r(t))u(t), \ \forall\, t \geq t_0, \quad (5.17)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ are state and input, respectively. $\{x(\theta) : t_0 - 2h \leq \theta \leq t_0\} = x_0 \in \mathscr{C}^b_{\mathscr{F}_0}([-2h, 0]; \mathbb{R}^n)$ and r_0 are initial function and initial Markovian regime, respectively. The control objective is to seek for a control law $u \in \mathscr{U}$ to minimize

$$J(t_0, x_0, u) = E\left\{ \int_{t_0}^{\infty} \{x^T(t)Q(t, r(t))x(t) + u^T(t)R(t, r(t))u(t)\} dt \Big| t_0, x_0, r_0 \right\}, \quad (5.18)$$

where for any $i \in \mathscr{S}$, matrices $Q(t, i)$ and $R(t, i)$ are positive semi-definite and positive definite, respectively.

Corollary 5.1 *For optimal control problem (5.17)–(5.18), if $r(t) = i$ at time t, then the optimal control law is*

$$
\begin{aligned}
u^0(t) = & -R^{-1}(t,i)D^T(t,i)\Lambda_1(t,i)x(t) \\
& - R^{-1}(t,i)D^T(t,i)\int_{-h}^{0}\Lambda_2(t,s,i)x(t+s)ds; \qquad (5.19)
\end{aligned}
$$

and the corresponding index value is

$$
\begin{aligned}
V(x(t_0),t_0,r_0) = & \, x^T(t_0)\Lambda_1(t_0,r_0)x(t_0) + 2x^T(t_0)\int_{-h}^{0}\Lambda_2(t_0,s,r_0)x(t_0+s)ds \\
& + \int_{-h}^{0}\int_{-h}^{0} x^T(t_0+s)\Lambda_3(t_0,r,s,r_0)x(t_0+r)\,dr\,ds, \qquad (5.20)
\end{aligned}
$$

where $\Lambda_1(t,i)$, $\Lambda_2(t,s,i)$, $\Lambda_3(t,r,s,i)$, $i \in \mathscr{S}$ are $n \times n$ matrices, $\Lambda_1(t,i)$ are symmetric positive definite matrices, $\Lambda_3(t,r,s,i) = \Lambda_3^T(t,r,s,i)$, $\Lambda_2(t,s,i)$, $\Lambda_3(t,r,s,i)$ are differentiable on t, r and s, and are the solutions of the following coupled equations,

$$
\begin{aligned}
& \dot{\Lambda}_1(t,i) + A^T(t,i)\Lambda_1(t,i) + \Lambda_1(t,i)A(t,i) + Q(t,i) + 2\Lambda_2(t,0,i) \\
& \quad - \Lambda_1(t,i)D(t,i)R^{-1}(t,i)D^T(t,i)\Lambda_1(t,i) + \sum_{j=1}^{N}\pi_{ij}\Lambda_1(t,j) = 0, \qquad (5.21) \\
& B^T(t,i)\Lambda_1(t,i) - \Lambda_2^T(t,-h,i) = 0, \qquad (5.22) \\
& A^T(t,i)\Lambda_2(t,s,i) + \frac{\partial\Lambda_2(t,s,i)}{\partial t} - \frac{\partial\Lambda_2(t,s,i)}{\partial s} \\
& \quad - \Lambda_1(t,i)D(t,i)R^{-1}(t,i)D^T(t,i)\Lambda_2(t,s,i) + \sum_{j=1}^{N}\pi_{ij}\Lambda_2(t,s,j) = 0, \qquad (5.23) \\
& B^T(t,i)\Lambda_2(t,s,i) - \Lambda_3(t,s,-h,i) = 0, \qquad (5.24) \\
& \frac{\partial\Lambda_3(t,r,s,i)}{\partial t} - \frac{\partial\Lambda_3(t,r,s,i)}{\partial r} - \frac{\partial\Lambda_3(t,r,s,i)}{\partial s} + \sum_{j=1}^{N}\pi_{ij}\Lambda_3(t,r,s,j) \\
& \quad - \Lambda_2^T(t,s,i)D(t,i)R^{-1}(t,i)D^T(t,i)\Lambda_2(t,r,i) = 0, \qquad (5.25)
\end{aligned}
$$

where $r,\ s \in [-h, 0]$.

Proof Let

$$V(x(t), t, i) = x^T(t)\Lambda_1(t, i)x(t) + 2x^T(t)\int_{-h}^{0}\Lambda_2(t, s, i)x(t+s)ds + \int_{-h}^{0}\int_{-h}^{0} x^T(t+s)\Lambda_3(t, r, s, i)x(t+r)\, dr\, ds,$$

for any $t \geq t_0$. Apply infinitesimal generator to $V(x(t), t, i)$ and add $x^T(t)Q(t, i)x(t) + u^T(t)R(t, i)u(t)$, we get

$$L(t, x, u, i) = \mathfrak{L}V(x, t, i) + x^T(t)Q(t, i)x(t) + u^T(t)R(t, i)u(t),$$

where

$$\begin{aligned}\mathfrak{L}V(x, t, i) = {} & x^T(t)[2A^T(t, i)\Lambda_1(t, i) + \dot{\Lambda}_1(t, i) + 2\Lambda_2(t, 0, i) + \sum_{j=1}^{N}\pi_{ij}\Lambda_1(t, j)]x(t) \\ & + x^T(t-h)[2B^T(t, i)\Lambda_1(t, i) - 2\Lambda_2^T(t, -h, i)]x(t) \\ & + x^T(t)\int_{-h}^{0}[2A^T(t, i)\Lambda_2(t, s, i) + 2\frac{\partial\Lambda_2(t, s, i)}{\partial t} - 2\frac{\partial\Lambda_2(t, s, i)}{\partial s} \\ & + \sum_{j=1}^{N}\pi_{ij}\Lambda_2(t, s, j)]x(t+s)ds + x^T(t-h)\int_{-h}^{0}[2B^T(t, i)\Lambda_2(t, s, i) \\ & - 2\Lambda_3(t, s, -h, i)]x(t+s)ds + \int_{-h}^{0}\int_{-h}^{0} x^T(t+s)[\frac{\partial\Lambda_3(t, r, s, i)}{\partial t} \\ & - \frac{\partial\Lambda_3(t, r, s, i)}{\partial r} - \frac{\partial\Lambda_3(t, r, s, i)}{\partial s} \\ & + \sum_{j=1}^{N}\pi_{ij}\Lambda_3(t, r, s, j)]x(t+r)dsdr + 2u^T(t)D^T(t, i)\Lambda_1(t, i)x(t) \\ & + 2u^T(t)D^T(t, i)\int_{-h}^{0}\Lambda_2(t, s, i)x(t+s)ds.\end{aligned}$$

By some simple calculations, under the control (5.19), we can get $\frac{\partial L(t,x,u,i)}{\partial u}|_{u=u^0} = 0$, $\frac{\partial^2 L(t,x,u,i)}{\partial u^2}|_{u=u^0} = 2R(t, i) > 0$. This verifies the condition (i) of Theorem 5.7.

Furthermore, under the conditions (5.21)–(5.25), we have

$$
\begin{aligned}
&L(t, x, u^0, i) \\
&= x^T(t)[2A^T(t,i)\Lambda_1(t,i) + Q(t,i) + \dot{p}_1(t,i) + 2\Lambda_2(t,0,i) + \sum_{j=1}^{N} \pi_{ij}\Lambda_1(t,j) \\
&- \Lambda_1(t,i)D(t,i)R^{-1}(t,i)D^T(t,i)\Lambda_1(t,i)]x(t) + x^T(t-h)[-2\Lambda_2^T(t,-h,i) \\
&+ 2B^T(t,i)\Lambda_1(t,i)]x(t) + x^T(t)\int_{-h}^{0}[2A^T(t,i)\Lambda_2(t,s,i) + 2\frac{\partial \Lambda_2(t,s,i)}{\partial t} \\
&+ \sum_{j=1}^{N} \pi_{ij}\Lambda_2(t,s,j) - 2\Lambda_1(t,i)D(t,i)R^{-1}(t,i)D^T(t,i)\Lambda_2(t,s,i) \\
&- 2\frac{\partial \Lambda_2(t,s,i)}{\partial s}]x(t+s)\,ds + x^T(t-h)\int_{-h}^{0}[2B^T(t,i)\Lambda_2(t,s,i) \\
&- 2\Lambda_3(t,s,-h,i)] \times x(t+s)ds + \int_{-h}^{0}\int_{-h}^{0} x^T(t+s)[\frac{\partial \Lambda_3(t,r,s,i)}{\partial t} \\
&- \frac{\partial \Lambda_3(t,r,s,i)}{\partial r} - \frac{\partial \Lambda_3(t,r,s,i)}{\partial s} - \Lambda_2^T(t,s,i)D(t,i)R^{-1}(t,i)D^T(t,i)\Lambda_2(t,r,i) \\
&+ \sum_{j=1}^{N} \pi_{ij}\Lambda_3(t,r,s,j)]x(t+r)dsdr \equiv 0, \forall i \in \mathscr{S}.
\end{aligned}
$$

So, the condition (ii) of Theorem 5.7 is true.

Similar to [2, 3], we can show the existence and uniqueness of the solution of the equations (5.21)–(5.25). Thus, the control (5.19) is well defined. Further, the existence and uniqueness of the solution process to the system (5.17) under the control (5.19) can be obtained directly from the Theorem 3.1 in [1]. This verifies the condition (iii) of Theorem 5.7.

We now verify the condition (iv) of Theorem 5.7. From

$$
\begin{aligned}
\dot{v} &= -G(t, v, x^0(t), r(t), u^0(t)) \\
&= \begin{cases} -[E\{x^0(t)\}]^T Q(t,r(t))E\{x^0(t)\} - [E\{u^0(t)\}]^T R(t,r(t))E\{u^0(t)\} < 0, & v > 0; \\ 0, & v = 0, \end{cases}
\end{aligned}
$$

we have $\lim_{t\to\infty} v(t, t_0, v_0, E\{x(t_0)\}, r_0) = 0,\ v_0 \geq 0$. Thus, the condition (iv) of Theorem 5.7 is true.

Therefore, by Theorem 5.7 and Remark 5.5, (5.19) is an optimal control law of the system (5.17), and (5.20) is the optimal control index value.

Remark 5.6 Theorem 5.7 and Corollary 5.1 consider the infinite time horizon case of optimal control of stochastic systems with jump parameters and time-delays. For the deterministic and finite time horizon case, it is referred to [2]. The results we have obtained generalize the work of [2, 14] to stochastic systems with Markovian jump parameters and time-delays.

Remark 5.7 Just as in [2, 13], we need to solve some coupled Riccati equations. To this end, we can refer to some recent research papers based on LMIs or Riccati equations, such as [7, 17–20] etc.

5.7 Summary

In this chapter, for stochastic nonlinear systems with both Markovian jump parameters and time-delays, some new definitions and criteria of practical stability (controllability) are given, which lay the foundation for our further study, such as the designing of the practical stabilization control law. Besides, a Hamilton-Jacobi-Bellman equation is obtained, which have been used to get the optimal control law and the optimal index value.

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Chapter 6
Networked Control System: A Markovian Jump System Approach

This chapter proposes a packet-based control approach to networked control systems. This approach takes advantage of the packet-based transmission of the network and as a consequence the control law can be designed with explicit compensation for the network-induced delay, data packet dropout and data packet disorder in both forward and backward channels. Under the Markov chain assumption of the network-induced delay (data packet dropout as well), the sufficient and necessary conditions for the stochastic stability and stabilization of the closed-loop system are obtained.

6.1 Introduction

Networked Control Systems (NCSs) are control systems whose control loop is closed via some form of communication network instead of connected directly as assumed in conventional control systems [7]. These communication networks include the control-oriented networks such as the control area network, DeviceNet, etc., but more and more data networks that are not specifically optimized for real-time control purpose, like the Internet, have now been popular in NCSs. As is known, a communication network inevitably introduces communication constraints to the control systems, e.g., network-induced delay, data packet dropout, data packet disorder, data rate constraint, etc. Despite the advantages of the remote and distribute control that NCSs brings, the aforementioned communication constraints in NCSs present a great challenge for conventional control theory [8, 10, 13, 14, 17, 19, 26].

The early work on NCSs has been done mainly from the control theory perspective. Such conventional control theories as time delay system theory [3, 18, 28], stochastic control theory [9, 11, 20, 23], switched system theory [12, 21, 27], have found their applications to NCSs by, typically speaking, modeling the communication network as one or several negative parameters (mostly a delay parameter) to the system, and then

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conventional methods in control theory can be used to design and analyse NCSs. In the recent years, the so-called “co-design" approach to NCSs becomes popular. This approach regards the design and analysis of NCSs as an inter-disciplinary problem at the boundary of control, communication and computation. Thus the consequent idea is to explore all the possible perspectives that may help the design and analysis of NCSs but is not limited to control itself [4, 6, 16, 24, 25]. The co-design principle has become one of the main streams in the future development of NCSs.

We here report a work on NCSs, within the co-design framework, by more effectively using the packet-based transmission in NCSs. This characteristic means that one data packet can encode multiple control signals, thus making it possible for us to send a sequence of forward control predictions simultaneously, impossible in the conventional system settings. Consequently by designing a comparison rule at the actuator side, the packet-based control approach can explicitly compensate for the communication constraints including the network-induced delay, data packet dropout and data packet disorder simultaneously in both forward and backward channels. This merit can not be achieved using conventional control approaches as in, e.g., [2, 23], where the characteristics of the network hves not been specially considered.

We model the characteristics of the round trip delay as Markovian, and then the closed-loop system is obtained as a Markovian jump system. Within the MJSs framework, the sufficient and necessary condition for the stochastic stability and stabilization of the closed-loop system with the packet-based control approach is obtained. This is an example showing how MJSs can be useful in the area of NCSs.

The remainder of the chapter is organized as follows. Section 6.2 presents the problem under consideration, followed by the design of the packet-based control approach in Sect. 6.3. For the derived closed-loop system, the stochastic stability and stabilization results are obtained in Sect. 6.4, which is then verified numerically in Sect. 6.5. Section 6.6 concludes the chapter.

6.2 Description of Networked Control Systems

The NCS setup considered is shown in Fig. 6.1, where $\tau_{sc,k}$ and $\tau_{ca,k}$ are the network-induced delays in the backward and forward channels (called “backward channel delay” and “forward channel delay” respectively hereafter) and the plant is linear in discrete-time, and can be represented by

$$x(k+1) = Ax(k) + Bu(k) \tag{6.1}$$

with $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. The full state information is assumed to be available.

It is noticed that the forward channel delay $\tau_{ca,k}$ is not available for the controller when the control action is calculated at time k, since $\tau_{ca,k}$ occurs after the determination of the control action, see Fig. 6.1. For this reason, when applying conventional design techniques such as those in time delay systems to NCSs, the active compen-

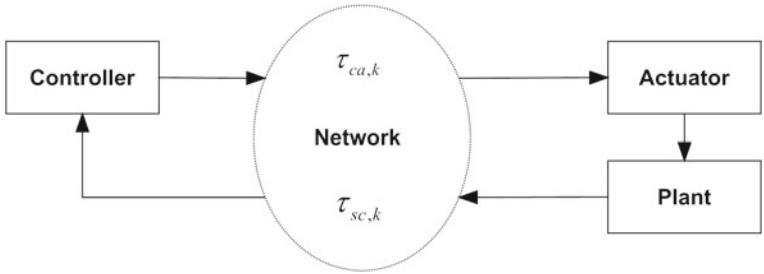

Fig. 6.1 The block diagram of a networked control system

sation for the forward channel delay can not be achieved. That is, the control law using conventional control approach to NCSs is typically obtained as

$$u(k) = Kx(k - \tau^*_{sc,k} - \tau^*_{ca,k}), \tag{6.2}$$

where $\tau^*_{sc,k}$ and $\tau^*_{ca,k}$ are the network-induced delays of the control action that is actually applied to the plant at time k and the feedback gain K is fixed for all network conditions. The fact that K is fixed implies that this conventional design technique is conservative in the networked control environment, since it loses the capability of actively compensating for the communication constraints while the system is up and running.

A packet-based control approach is therefore designed with explicit consideration of the communication constraints in NCSs, as detailed in the next section. The control law based on this approach is obtained as follows, when no time-synchronization among the control components is available (Algorithm 6.1),

$$u(k) = K(\tau^*_{sc,k}, \tau^*_{ca,k})x(k - \tau^*_{sc,k} - \tau^*_{ca,k}), \tag{6.3}$$

when with the time-synchronization (Algorithm 6.2), it is obtained as

$$u(k) = K(\tau^*_k)x(k - \tau^*_k), \tag{6.4}$$

where $\tau^*_k = \tau^*_{sc,k} + \tau^*_{ca,k}$. It is noted that using the control laws in (6.3) and (6.4), the feedback gains can be designed with explicit consideration of the communication constraints, thus enabling us to actively compensate for the communication constraints in NCSs by applying different feedback gains for different network conditions, as is done in Sect. 6.4. In the following remark, we notice that other researchers have also attempted to achieve such an advantage which however is not realizable in practice since no supportive design method has been given.

6.3 Packet-Based Control for NCSs

For the design of the packet-based control approach for NCSs, the following assumptions are required.

Assumption 6.1 *The controller and the actuator (plant) are time-synchronized and the data packets sent from both the sensor and the controller are time-stamped.*

Assumption 6.2 *The sum of the maximum forward (backward) channel delay and the maximum number of consecutive data packet dropout (disorder as well) is upper bounded by $\bar{\tau}_{ca}$ ($\bar{\tau}_{sc}$ accordingly) and*

$$\bar{\tau}_{ca} \leq \frac{B_p}{B_c} - 1, \tag{6.5}$$

where B_p is the size of the effective load of the data packet and B_c is the bits required to encode a single step control signal.

Remark 6.1 Time-synchronization is required for the implementation of the control law in (6.3), which can be relaxed for the control law in (6.4), see Remark 6.4. With time-synchronization among the control components and the time stamps used, the network-induced delay that each data packet experiences can then be known by the controller and the actuator upon its arrival.

Remark 6.2 In Assumption 6.2, the upper bound of the delay and dropout is only meant for those received successfully; A dropped data packet is not treated as an infinite delay. In light of the UDP (User Datagram Protocol) that is widely used in NCSs, this upper bound assumption is thus reasonable in practice as well as necessary in theory. Furthermore, the constraint in (6.5) is easy to be satisfied, e.g., $B_p = 368\,bit$ for Ethernet IEEE 802.3 frame which is often used [15], while an 8-bit data (i.e., $B_c = 8\,bit$) can encode $2^8 = 256$ different control actions which is ample for most control implementations; In this case, 45 steps of forward channel delay is allowed by (6.5) which can actually meet the requirements of most practical control systems.

The block diagram of the packet-based control structure is illustrated in Fig. 6.2. It is distinct from the conventional control structure in two respects: the specially designed packet-based controller and the corresponding Control Action Selector (CAS) at the actuator side.

In order to implement the control laws in (6.3) and (6.4), we take advantage of the packet-based transmission of the network to design a packet-based controller instead of trying to obtain directly the current forward channel delay as this is actually impossible in practice. As for the control law in (6.3), the packet-based controller determines a sequence of forward control actions as follows and sends them together in one data packet to the actuator,

$$U_1(k|k-\tau_{sc,k}) = [u(k|k-\tau_{sc,k}) \ \ldots \ u(k+\bar{\tau}_{ca}|k-\tau_{sc,k})]^T, \tag{6.6}$$

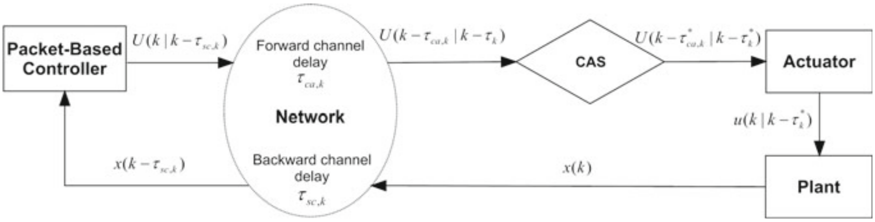

Fig. 6.2 Packet-based control for networked control systems

where $u(k+i|k-\tau_{sc,k}), i = 0, 1, \ldots, \tau_{ca,k}$ are the forward control action predictions based on information up to time $k - \tau_{sc,k}$.

When a data packet arrives at the actuator, the designed CAS compares its time stamp with the one already in CAS and only the one with the latest time stamp is saved. Denote the forward control sequence already in CAS and the one just arrived by $U_1(k_1 - \tau_{ca,k_1}|k_1 - \tau_{k_1})$ and $U_1(k_2 - \tau_{ca,k_2}|k_2 - \tau_{k_2})$ respectively, then the chosen sequence is determined by the following comparison rule,

$$U_1(k - \tau^*_{ca,k}|k - \tau^*_k) = \begin{cases} U_1(k_2 - \tau_{ca,k_2}|k_2 - \tau_{k_2}), & \text{if } k_1 - \tau_{k_1} < k_2 - \tau_{k_2}; \\ U_1(k_1 - \tau_{ca,k_1}|k_1 - \tau_{k_1}), & \text{otherwise.} \end{cases} \quad (6.7)$$

The comparison process is introduced because different data packets may experience different delays thus producing a situation where a packet sent earlier may arrive at the actuator later, that is, data packet disorder. After the comparison process, only the latest available information is used.

CAS also determines the appropriate control action from the forward control sequence $U_1(k - \tau^*_{ca,k}|k - \tau^*_k)$ at each time instant as follows

$$u(k) = u(k|k - \tau^*_{sc,k} - \tau^*_{ca,k}). \quad (6.8)$$

It is necessary to point out that the appropriate control action determined by (6.8) is always available provided Assumption 6.2 holds and (6.8) is equivalent to the control law in (6.3) if state feedback is used, i.e.,

$$u(k) = u(k|k - \tau^*_{sc,k} - \tau^*_{ca,k}) = K(\tau^*_{sc,k}, \tau^*_{ca,k})x(k - \tau^*_{sc,k} - \tau^*_{ca,k}). \quad (6.9)$$

The packet-based control algorithm with the control law in (6.3) can now be summarized as follows based on Assumptions 6.1 and 6.2.

Algorithm 6.1 *Packet-based control with the control law in (6.3)*

Step1. At time k, if the packet-based controller receives the delayed state data $x(k - \tau_{sc,k})$, then, it

- Reads current backward channel delay $\tau_{sc,k}$;
- Calculates the forward control sequence as in (6.6);
- Packs $U_1(k|k - \tau_{sc,k})$ and sends it to the actuator in one data packet with time stamps k and $\tau_{sc,k}$.

If no data packet is received at time k, then let $k = k + 1$ and wait for the next time instant.

Step2. CAS updates its forward control sequence by (6.7) once a data packet arrives;
Step3. The control action in (6.9) is picked out from CAS and applied to the plant.

In practice, it is often the case that we do not need to identify separately the forward and backward channel delays since it is normally the round trip delay that affects the system performance. In such a case, the simpler control law in (6.4) instead of that in (6.3) is applied, for which the following assumption is required instead of Assumption 6.2.

Assumption 6.3 *The sum of the maximum network-induced delay and the maximum number of continuous data packet dropout in the round trip is upper bounded by $\bar{\tau}$ and*

$$\bar{\tau} \leq \frac{B_p}{B_c} - 1. \tag{6.10}$$

With the above assumption, the packet-based controller is modified as follows

$$U_2(k|k - \tau_{sc,k}) = [u(k - \tau_{sc,k}|k - \tau_{sc,k}) \ \dots \ u(k - \tau_{sc,k} + \bar{\tau}|k - \tau_{sc,k})]^T. \tag{6.11}$$

It is noticed that in such a case the backward channel delay $\tau_{sc,k}$ is not required for the controller, since the controller simply produces $(\bar{\tau} + 1)$ step forward control actions whenever a data packet containing sensing data arrives. This relaxation implies that the time-synchronization between the controller and the actuator (plant) is not required and thus Assumption 6.1 can then be modified as follows.

Assumption 6.4 *The data packets sent from the sensor are time-stamped.*

The comparison rule in (6.7) and the determination of the actual control action in (6.9) remain unchanged since both of them are based on the round trip delay τ_k and in this case the control law with state feedback is obtained as follows, as presented in (6.4),

$$u(k) = u(k|k - \tau_k^*) = K(\tau_k^*)x(k - \tau_k^*). \tag{6.12}$$

The packet-based control algorithm with the control law in (6.4) can now be summarized as follows based on Assumptions 6.3 and 6.4.

Algorithm 6.2 *Packet-based control with the control law in (6.4)*

Step1. At time k, if the packet-based controller receives the delayed state data $x(k-\tau_{sc,k})$, then,

- Calculates the forward control sequence as in (6.11);
- Packs $U_2(k|k-\tau_{sc,k})$ and sends it to the actuator in one data packet.

If no data packet is received at time k, then let $k=k+1$ and wait for the next time instant.

Step2-Step3. remain the same as in Algorithm 6.1.

Remark 6.3 From the design procedure of the packet-based control approach it is seen that the implementation of this approach requires only: (1) a modified controller to produce the sequences of the forward control signals in (6.6) or (6.11) and (2) the so designed CAS at the actuator side to compensate for the communication constraints. In practice the latter could be a separate control component added to the system, and the packet-based controller can be designed using any appropriate methods that can give rise to a good system performance. Therefore, this approach can be readily implemented in practice. Furthermore, the fact that conventional control design methods can still be fitted in the packet-based control framework also makes the proposed approach a universal solution to NCSs.

6.4 Stochastic Modeling and Stabilization

It is noticed that the control law in (6.3) equals that in (6.4) if $K(\tau_k^*)=K(\tau_{sc,k}^*,\tau_{ca,k}^*)$ which is generally true in practice. Thus for simplicity only the closed-loop system with the control law in (6.4) (i.e., Algorithm 6.2) is analyzed.

Let $X(k)=[x^T(k)\,x^T(k-1)\,\cdots\,x^T(k-\bar{\tau})]^T$, then the closed-loop system with the control law in (6.4) can be written as

$$X(k+1)=\Xi(\tau_k^*)X(k), \tag{6.13}$$

where $\Xi(\tau_k^*)=\begin{pmatrix} A & \cdots & BK(\tau_k^*) & \cdots & \cdots \\ I_n & & & & 0 \\ & I_n & & & 0 \\ & & \ddots & & \vdots \\ & & & I_n & 0 \end{pmatrix}$ and I_n is the identity matrix with rank n.

6.4.1 The MJS Model of the Packet-Based Control Approach for NCSs

In NCSs, it is reasonable to model the round trip delay $\{\tau_k; k = 0, 1, \ldots\}$ as a homogeneous ergodic Markov chain [23]. Here in order to take explicit account of the data packet dropout, Markov chain $\{\tau_k; k = 0, 1, \ldots\}$ is assumed to take values from $\mathscr{M} = \{0, 1, 2, \ldots, \bar{\tau}, \infty\}$ where $\tau_k = 0$ means no delay in round trip while $\tau_k = \infty$ implies a data packet dropout in either the backward or the forward channel. Let the transition probability matrix of $\{\tau_k; k = 0, 1, \ldots\}$ be denoted by $\Lambda = [\lambda_{ij}]$ where

$$\lambda_{ij} = P\{\tau_{k+1} = j|\tau_k = i\}, i, j \in \mathscr{M},$$

$P\{\tau_{k+1} = j|\tau_k = i\}$ is the probability of τ_k jumping from state i to j, $\lambda_{ij} \geq 0$ and

$$\sum_{j\in\mathscr{M}} \lambda_{ij} = 1, \forall i, j \in \mathscr{M}.$$

The initial distribution of $\{\tau_k; k = 0, 1, \ldots\}$ is defined by

$$P\{\tau_0 = i\} = p_i, i \in \mathscr{M}.$$

According to the comparison rule in (6.7), the round trip delay of the control actions that are actually applied to the plant can be determined by the following equation.

$$\tau^*_{k+1} = \begin{cases} \tau^*_k + 1, \text{ if } \tau_{k+1} > \tau^*_k; \\ \tau^*_k - r, \text{ if } \tau^*_k - r = \tau_{k+1} \leq \tau^*_k. \end{cases} \tag{6.14}$$

Remark 6.4 The data packet dropout is explicitly considered by including the state $\tau_k = \infty$ into the state space Λ; The data packet disorder is also considered by (6.14): In our stochastic model the network-induced delay, data packet dropout and data packet disorder are all considered simultaneously. To the best knowledge of the authors, there is no analogous analysis available in the literature to date.

Lemma 6.1 $\{\tau^*_k; k = 0, 1, \ldots\}$ *is a non-homogeneous Markov chain with state space* $\mathscr{M}^* = \{0, 1, 2, \ldots, \bar{\tau}\}$ *whose transition probability matrix* $\Lambda^*(k) = [\lambda^*_{ij}(k)]$ *is defined by*

$$\lambda^*_{ij}(k) = \begin{cases} \dfrac{\sum\limits_{l_1\in\mathscr{M}, l_1\geq i} \pi_{l_1}(k)\lambda_{l_1 j}}{\sum\limits_{l_1\in\mathscr{M}, l_1\geq i} \pi_{l_1}(k)}, j \leq i; \\ \dfrac{\sum\limits_{l_1\in\mathscr{M}, l_1\geq i}\sum\limits_{l_2\in\mathscr{M}, l_2> i} \pi_{l_1}(k)\lambda_{l_1 l_2}}{\sum\limits_{l_1\in\mathscr{M}, l_1\geq i} \pi_{l_1}(k)}, j = i + 1; \\ 0, \ otherwise. \end{cases} \tag{6.15}$$

where $\pi_j(k) = \sum_{i\in\mathscr{M}} p_i\lambda_{ij}^{(k)}$ *and* $\lambda_{ij}^{(k)}$ *is the k-step transition probability of* τ_k *from state i to j.*

Proof The comparison rule in (6.14) implies that the probability event $\{\tau_k^* = i\} \in \sigma(\tau_k, \tau_{k-1}, \ldots, \tau_1, \tau_0)$. Thus it is readily concluded that τ_k^* is also a Markov chain since τ_k as a Markov chain evolves independently. It is obvious that τ_k^* can not be ∞ and thus its state space is $\mathscr{M}^* = \{0, 1, 2, \ldots, \bar{\tau}\}$. Furthermore, noticing $\{\tau_k^* = i\} = \{\tau_{k-1}^* = i-1, \tau_k > i-1\} \cup \{\tau_{k-1}^* \geq i, \tau_k = i\}$ we have

1. If $j \leq i$, then

$$P\{\tau_{k+1}^* = j|\tau_k^* = i\} = P\{\tau_{k+1} = j|\tau_k^* = i\} = P\{\tau_{k+1} = j|\tau_k \geq i\}$$
$$= \frac{\sum_{l_1\in\mathscr{M}, l_1\geq i} \pi_{l_1}(k)\lambda_{l_1 j}}{\sum_{l_1\in\mathscr{M}, l_1\geq i} \pi_{l_1}(k)}$$

2. If $j = i+1$, then

$$P\{\tau_{k+1}^* = j|\tau_k^* = i\} = P\{\tau_{k+1} > i|\tau_k^* = i\} = P\{\tau_{k+1} > i|\tau_k \geq i\}$$
$$= \frac{\sum_{l_1\in\mathscr{M}, l_1\geq i} \sum_{l_2\in\mathscr{M}, l_2> i} \pi_{l_1}(k)\lambda_{l_1 l_2}}{\sum_{l_1\in\mathscr{M}, l_1\geq i} \pi_{l_1}(k)}$$

which completes the proof.

The following well-known result for homogeneous ergodic Markov chains [1] is required for the stochastic stability analysis in this section.

Lemma 6.2 *For the homogeneous ergodic Markov chain* $\{\tau_k; k = 0, 1, \ldots\}$ *with any initial distribution, there exists a limit probability distribution* $\pi = \{\pi_i; \pi_i > 0, i \in \mathscr{M}\}$ *such that for each* $j \in \mathscr{M}$,

$$\sum_{i\in\mathscr{M}} \lambda_{ij}\pi_i = \pi_j, \quad \sum_{i\in\mathscr{M}} \pi_i = 1 \tag{6.16}$$

and

$$|\pi_i(k) - \pi_i| \leq \eta\xi^k \tag{6.17}$$

for some $\eta \geq 0$ *and* $0 < \xi < 1$.

Proposition 6.1 *For* N_1 *that is large enough and some nonzero* η^* *the following inequality holds*

$$|\lambda_{ij}^*(k) - \lambda_{ij}^*| \leq \eta^*\xi^k, \; k > N_1, \tag{6.18}$$

where $\Lambda^* = [\lambda^*_{ij}]$ *with*

$$\lambda^*_{ij} = \begin{cases} \dfrac{\sum_{l_1\in\mathscr{M}, l_1\ge i} \pi_{l_1}\lambda_{l_1 j}}{\sum_{l_1\in\mathscr{M}, l_1\ge i} \pi_{l_1}}, \ \mathit{if}\ j \le i; \\ \dfrac{\sum_{l_1\in\mathscr{M}, l_1\ge i}\sum_{l_2\in\mathscr{M}, l_2> i} \pi_{l_1}\lambda_{l_1 l_2}}{\sum_{l_1\in\mathscr{M}, l_1\ge i} \pi_{l_1}}, \ \mathit{if}\ j = i+1; \\ 0,\ \mathit{otherwise}. \end{cases} \tag{6.19}$$

Proof It can be readily obtained from (6.15), (6.17) and (6.19).

6.4.2 Stochastic Stability and Stabilization

The following definition of stochastic stability is used.

Definition 6.1 The closed-loop system in (6.13) is said to be stochastically stable if for every finite $X_0 = X(0)$ and initial state $\tau^*_0 = \tau^*(0) \in \mathscr{M}$, there exists a finite $W > 0$ such that the following inequality holds,

$$E\{\sum_{k=0}^{\infty} ||X(k)||^2 | X_0, \tau^*_0\} < X_0^T W X_0, \tag{6.20}$$

where $E\{X\}$ is the expectation of the random variable X.

Theorem 6.1 *The closed-loop system in (6.13) is stochastically stable if and only if there exists* $P(i) > 0, i \in \mathscr{M}^*$ *such that the following* $(\bar{\tau}+1)$ *LMIs hold*

$$L(i) = \sum_{j\in\mathscr{M}^*} \lambda^*_{ij}\Xi^T(j)P(j)\Xi(j) - P(i) < 0, \forall i \in \mathscr{M}^*. \tag{6.21}$$

Proof Sufficiency. For the closed-loop system in (6.13), consider the following quadratic function given by

$$V(X(k), k) = X^T(k)P(\tau^*_k)X(k). \tag{6.22}$$

We have

$$\begin{aligned} E\{\Delta V(X(k), k)\} &= E\{X^T(k+1)P(\tau^*_{k+1})X(k+1)|X(k), \tau^*_k = i\} - X^T(k)P(i)X(k) \\ &= \sum_{j\in\mathscr{M}^*} \lambda^*_{ij}(k+1)X^T(k)\Xi^T(j)P(j)\Xi(j)X(k) - X^T(k)P(i)X(k) \\ &= X^T(k)[\sum_{j\in\mathscr{M}^*} \lambda^*_{ij}(k+1)\Xi^T(j)P(j)\Xi(j) - P(i)]X(k). \end{aligned}$$

From condition (6.21) we obtain

$$X^T(k)[\sum_{j\in\mathscr{M}^*}\lambda^*_{ij}\Xi^T(j)P(j)\Xi(j)-P(i)]X(k)\le-\lambda_{\min}(-L(i))X^T(k)X(k)$$
$$\le-\beta||X(k)||^2, \tag{6.23}$$

where $\beta=\inf\{\lambda_{\min}(-L(i)); i\in\mathscr{M}^*\}>0$. Thus for $k>N_1$,

$$\begin{aligned}E\{\Delta V(X(k),k)\}&=X^T(k)[\sum_{j\in\mathscr{M}^*}\lambda^*_{ij}(k+1)\Xi^T(j)P(j)\Xi(j)-P(i)]X(k)\\&\le X^T(k)[\sum_{j\in\mathscr{M}^*}\lambda^*_{ij}\Xi^T(j)P(j)\Xi(j)-P(i)]X(k)\\&\quad+X^T(k)\sum_{j\in\mathscr{M}^*}|\lambda^*_{ij}(k+1)-\lambda^*_{ij}|\Xi^T(j)P(j)\Xi(j)X(k)\\&\le-\beta||X(k)||^2+\eta^*\xi^{k+1}X^T(k)\sum_{j\in\mathscr{M}^*}\Xi^T(j)P(j)\Xi(j)X(k)\\&\le(\alpha\eta^*\xi^{k+1}-\beta)||X(k)||^2,\end{aligned}$$

where $\alpha=\sup\{\lambda_{\max}(\Xi^T(j)P(j)\Xi(j)); j\in\mathscr{M}^*\}>0$. Let $N_2=\inf\{M; M\in\mathbb{N}^+, M>\max\{N_1,\log_\xi\frac{\beta}{\alpha\eta^*}-1\}\}$. Then we have for $k\ge N_2$

$$E\{\Delta V(X(k),k)\}\le-\beta^*||X(k)||^2 \tag{6.24}$$

where $\beta^*=\beta-\alpha\eta^*\xi^{N_2+1}>0$. Summing from N_2 to $N>N_2$ we obtain

$$\begin{aligned}E\{\sum_{k=N_2}^{N}||X(k)||^2\}&\le\frac{1}{\beta^*}(E\{V(X(N_2),N_2)\}-E\{V(X(N+1),N+1)\})\\&\le\frac{1}{\beta^*}E\{V(X(N_2),N_2)\},\end{aligned}$$

which implies that

$$E\{\sum_{k=0}^{\infty}||X(k)||^2\}\le\frac{1}{\beta^*}E\{V(X(N_2),N_2)\}+E\{\sum_{k=0}^{N_2-1}||X(k)||^2\}. \tag{6.25}$$

This proves the stochastic stability of the closed-loop system in (6.13) by Definition 6.1.

Necessity. Suppose the closed-loop system in (6.13) is stochastically stable, that is,

$$E\{\sum_{k=0}^{\infty}||X(k)||^2|X_0, \tau_0^*\} < X_0^T W X_0. \tag{6.26}$$

Define

$$X^T(n)\bar{P}(N-n, \tau_n^*)X(n) = E\{\sum_{k=n}^{N} X^T(k)Q(\tau_k^*)X(k)|X_n, \tau_n^*\} \tag{6.27}$$

with $Q(\tau_k^*) > 0$. It is noticed that $X^T(n)\bar{P}(N-n, \tau_n^*)X(n)$ is upper bounded from (6.26) and monotonically non-decreasing as N increases since $Q(\tau_k^*) > 0$. Therefore its limit exists which is denoted by

$$X^T(n)P(i)X(n) = \lim_{N\to\infty} X^T(n)\bar{P}(N-n, \tau_n^* = i)X(n). \tag{6.28}$$

Since (6.28) is valid for any $X(n)$, we obtain

$$P(i) = \lim_{N\to\infty} \bar{P}(N-n, \tau_n^* = i) > 0. \tag{6.29}$$

Now consider

$$\begin{aligned} &E\{X^T(n)\bar{P}(N-n, \tau_n^*)X(n) - X^T(n+1)\bar{P}(N-n-1, \tau_{n+1}^*)X(n+1)|X_n, \tau_n^* = i\} \\ =&X^T(n)[\bar{P}(N-n, i) - \sum_{j\in\mathscr{M}^*} \lambda_{ij}^*(n+1)\Xi^T(j)\bar{P}(N-n-1, j)\Xi(j)]X(n) \\ =&X^T(n)Q(i)X(n). \end{aligned} \tag{6.30}$$

Since (6.30) is valid for any $X(n)$, we obtain

$$\bar{P}(N-n, i) - \sum_{j\in\mathscr{M}^*} \lambda_{ij}^*(n+1)\Xi^T(j)\bar{P}(N-n-1, j)\Xi(j) = Q(i) > 0. \tag{6.31}$$

Let $N \to \infty$,

$$P(i) - \sum_{j\in\mathscr{M}^*} \lambda_{ij}^*(n+1)\Xi^T(j)P(j)\Xi(j) > 0, \forall n$$

Let $n \to \infty$,

$$P(i) - \sum_{j\in\mathscr{M}^*} \lambda_{ij}^*\Xi^T(j)P(i)\Xi(j) > 0,$$

which completes the proof.

The result below readily follows using the Schur complement.

Corollary 6.1 *System (6.1) is stochastically stabilizable using the packet-based control approach with the control law in (6.4) if and only if there exist* $P(i) > 0$, $Z(i) > 0$, $K(i)$, $i \in \mathscr{M}^*$ *such that the following* $(\bar{\tau}+1)$ *LMIs hold*

$$\begin{pmatrix} P(i) & R(i) \\ R^T(i) & Q \end{pmatrix} > 0, \ i \in \mathscr{M}^* \tag{6.32}$$

with the equation constraints

$$P(i)Z(i) = I, \forall i \in \mathscr{M}^*, \tag{6.33}$$

where $R(i) = [(\lambda^*_{i0})^{\frac{1}{2}} \Xi^T(0) \ldots (\lambda^*_{i\bar{\tau}})^{\frac{1}{2}} \Xi^T(\bar{\tau})]$, $Q = diag\{Z(0) \ldots Z(\bar{\tau})\}$ *and* $\Xi(i)$ *(consequently* $K(i)$*) is defined in (6.13).*

The LMIs in Corollary 6.1 with the matrix inverse constraints in (6.33) can be solved using the Cone Complementarity Linearization (CCL) algorithm [5].

6.5 Numerical Simulation

A numerical example is considered in this section to illustrate the effectiveness of the propose approach. Consider the system in (6.1) with the following system matrices borrowed from [23],

$$A = \begin{pmatrix} 1.0000 & 0.1000 & -0.0166 & -0.0005 \\ 0 & 1.0000 & -0.3374 & -0.0166 \\ 0 & 0 & 1.0996 & 0.1033 \\ 0 & 0 & 2.0247 & 1.0996 \end{pmatrix}, B = \begin{pmatrix} 0.0045 \\ 0.0896 \\ -0.0068 \\ -0.1377 \end{pmatrix}.$$

This system is open-loop unstable with the eigenvalues at 1, 1, 1.5569 and 0.6423, respectively. In the simulation, the random round trip delay is upper bounded by 4, i.e., $\tau_k \in \mathscr{M} = \{0, 1, 2, 3, 4, \infty\}$, with the following transition probability matrix,

$$\Lambda = \begin{pmatrix} 0.1 & 0.2 & 0.2 & 0.3 & 0.2 & 0 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.1 & 0.1 \\ 0.24 & 0.06 & 0.48 & 0.12 & 0.1 & 0 \\ 0.15 & 0.25 & 0.3 & 0.15 & 0.1 & 0.05 \\ 0.3 & 0.3 & 0.2 & 0.1 & 0.1 & 0 \\ 0.3 & 0.3 & 0.15 & 0.15 & 0.1 & 0 \end{pmatrix}.$$

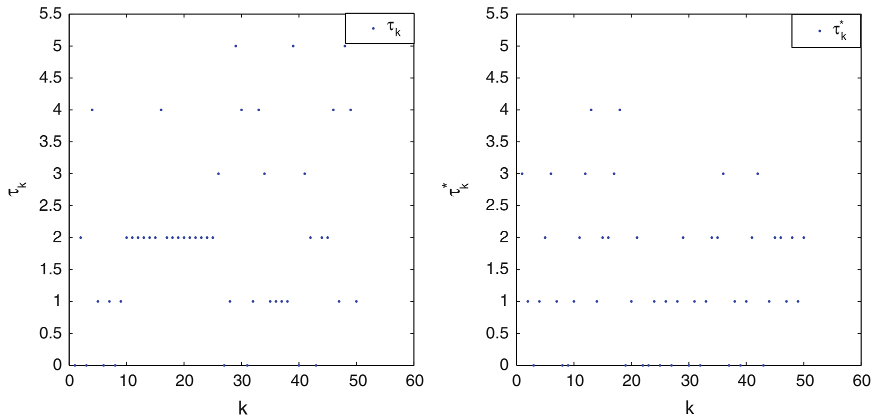

Fig. 6.3 Comparison of the practical delays τ_k and those after the comparison process τ_k^* where 5 on the vertical axis represents a data packet dropout

The limit distribution of the above ergodic Markov chain can be simply obtained by Lemma 6.2,

$$\pi = \begin{pmatrix} 0.1982 & 0.1814 & 0.3000 & 0.1738 & 0.1198 & 0.0268 \end{pmatrix}.$$

Λ^* in Proposition 6.1 can then be calculated by (6.19) as

$$\Lambda^* = \begin{pmatrix} 0.1982 & 0.8018 & 0 & 0 & 0 \\ 0.2224 & 0.1767 & 0.6008 & 0 & 0 \\ 0.2290 & 0.1699 & 0.3612 & 0.2398 & 0 \\ 0.2186 & 0.2729 & 0.2501 & 0.1313 & 0.1271 \\ 0.3000 & 0.3000 & 0.1909 & 0.1091 & 0.1000 \end{pmatrix}.$$

The comparison between the practical delays τ_k and those after the comparison process using the packet-based control approach τ_k^* is illustrated in Fig. 6.3 where 5 on the vertical axis represents a data packet dropout. From Fig. 6.3 it is seen that data packet dropout has been effectively dealt with using the packet-based control approach, by noticing that $\tau_k^* \in \mathscr{M}^* = \{0, 1, 2, 3, 4\}$.

From Corollary 6.1, the packet-based controller is obtained as follows, where it is seen that for different network conditions, different feedback gains are designed,

$$\begin{aligned} K(0) &= \begin{pmatrix} 0.5292 & 0.6489 & 22.4115 & 2.8205 \end{pmatrix}, \\ K(1) &= \begin{pmatrix} 0.3792 & 0.8912 & 20.2425 & 5.3681 \end{pmatrix}, \\ K(2) &= \begin{pmatrix} 0.0499 & 0.4266 & 15.6574 & 5.7322 \end{pmatrix}, \\ K(3) &= \begin{pmatrix} -0.4400 & -0.3003 & 9.2976 & 5.0540 \end{pmatrix}, \\ K(4) &= \begin{pmatrix} -0.8400 & -1.3422 & 2.7723 & 2.9173 \end{pmatrix}. \end{aligned}$$

Using the packet-based control approach with the above packet-based controller, the state trajectories of the closed-loop system is illustrated in Fig. 6.4 with the initial states $x(-3) = x(-2) = x(-1) = x(0) = [0\ 0.1\ 0\ \ -0.1]^T$, which demonstrates the stochastic stability of the closed-loop system.

On the contrary, without the packet-based control strategy, even using the same controller design method (that is, using $K(i) \equiv K(0), i \in \mathscr{M}$, i.e., $K(0)$ fixed for all network conditions), the system is shown to be unstable under the same simulation conditions, see Fig. 6.5. Furthermore, consider the conventional control approach proposed in [22] where no packet-based control structure was considered and the feedback gain was designed as $K = [0.9844\ 1.6630\ 25.9053\ 6.1679]$ fixed for all network conditions, the system is also shown to be unstable under the same simulation

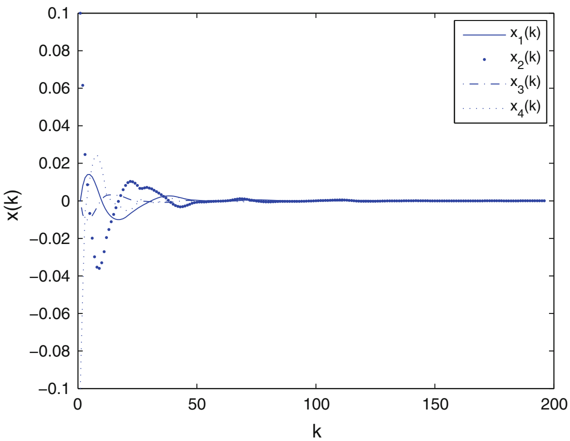

Fig. 6.4 The system is stable using the packet-based control approach

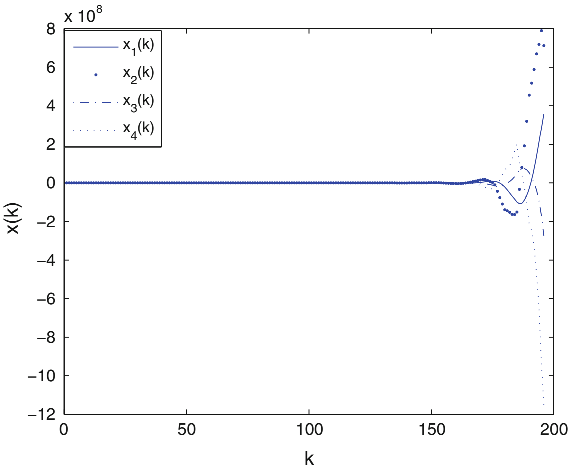

Fig. 6.5 The system is unstable without the packet-based control strategy, using $K(0)$ fixed for all network conditions

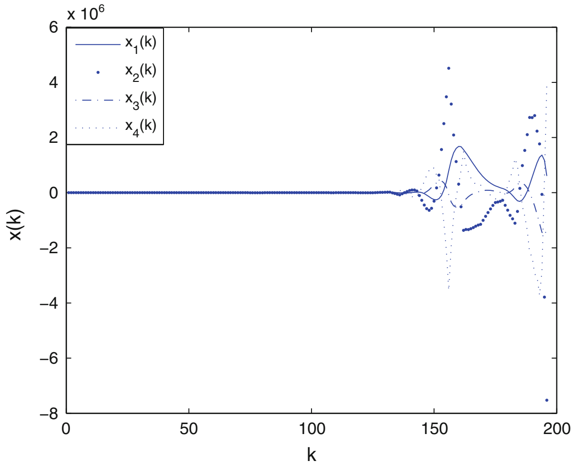

Fig. 6.6 The system is unstable using conventional control approach with a fixed feedback gain

conditions, see Fig. 6.6. These comparisons proves the effectiveness of the proposed packet-based control approach and the stabilized controller design method.

6.6 Summary

By taking advantage of the packet-based data transmission in NCSs, a packet-based control approach is proposed for NCSs, which can be used to actively compensate for the communication constraints in NCSs including network-induced delay, data packet dropout and data packet disorder simultaneously. The novel model obtained based on this approach offers the designers the freedom of designing different controllers for different network conditions. The stochastic stabilization result is then obtained by modeling the communication constraints as a homogeneous ergodic Markov chain and then the closed-loop system as a Markovian jump system. This result is based on a better understanding of the packet-based data transmission in the stochastic fashion and enabled the proposed packet-based control approach to be applied in practice.

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Chapter 7
Applications Based on the Markov Jump Theory

This chapter consists of two applications of Markovian jump systems. Section 7.2 considers the fault-tolerant control for wheeled mobile manipulators. We are concerned with the output feedback H_∞ control based on a high-gain observer for wheeled mobile manipulators, since the velocity signals are generally not available and indirectly obtained from the measured positions. We are to design a mode-dependent dynamic output feedback controller for wheeled mobile manipulators which guarantees not only the robust stochastic stability but also a prescribed disturbance attenuation level for the resulting closed-loop system, irrespective of the transition rate uncertainties. Section 7.3 considers the jump linear quadratic regulator problem of MJLS. A two-level regulating approach is employed to design the control law and the transition rate control policy. The problem of tuning the existing policy with respect to a prescribed quadratic performance criterion is formulated as a gradient projection based iterative optimization. Based on this method, a new policy is obtained with better performance than that of the initial policy.

7.1 Introduction

Wheeled mobile manipulators have attracted a lot of attention recently [30, 33]. However, besides exogenous disturbances which may increase the difficulty of reference tracking control for mobile manipulators, actuator failures (either in wheels or joints) might suddenly occur during the motion of mobile manipulators. The failed actuators, where the torque supplied to the motors of one or more joints vanishes suddenly, can destabilize the system with the possibility of damaging the robot components. When a free torque fault occurs, the fully actuated manipulator would become an under-actuated one, to avoid the necessity of stopping the robot when a fault occurs, the Markovian jump linear system (MJLS) theory was developed to design a procedure to incorporate abrupt changes in the manipulator configuration.

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Continuous-time MJLS [6, 33] is a hybrid system, which consists of a finite number of subsystems and a jumping law governing the switching among them. The jumping law, usually denote by $r(t)$, is a continuous-time Markov chain representing the activated subsystem at time t, i.e. the mode of the hybrid system. The subsystem can often be represented by differential equations which determine the evolution of the physical states, usually denote by $x(t)$, when the system mode is given. That is the evolution of the system states depends not only on each subsystem but also the jumping law. MJLS is widely used to model and analyze the practical systems subject to abrupt changes, such as component failures, sudden environmental disturbances and the abrupt variation of the operation point and the like.

MJLS method used to model and analyze fault occurrence for robotic systems is an effective but challenging work. In [37, 38], the proposed control based on state-feedback Markovian H_∞ control was proposed for fault-tolerant of three-link robotic manipulator. However, the wheeled mobile manipulators are obviously different from robotic manipulators due to nonholonomic constraints. Apparently, the existing control method [37, 38] for robotic manipulators is not suitable for the robots with velocities constraints. In this Chapter, we develop the methodology via Markovian control theory to evaluate fault tolerant mobile manipulators. First, the controller designed in this Chapter is H_∞ state-feedback, which requires that all the variables could be directly measured. However, it's generally not available for the mobile manipulators. To overcome this practical difficulty, we are concerned with the output feedback H_∞ control based on a high-gain observer. Second, for the reason that only the estimated values of the mode transition rates are available, and the estimation errors, referred to as switching probability uncertainties, may lead to instability or at least degraded performance of a system as the uncertainties in system matrices do [44]. In this part, two different types of descriptions about uncertain switching probabilities have been considered. The first one is the polygon description where the mode transition rate matrix is assumed to be in a convex hull with known vertices [13]. The other type is described in an element-wise way. In this case, the elements of the mode transition rate matrix are measured in practice while the error bounds are given [10]. In many situations, the element-wise uncertainty description can be more convenient as well as natural. In this Chapter, we consider the element-wise uncertainties in the mode transition rate matrix and based on this we give a more realistic Markovian model for the mobile manipulator system. The uncertainties are allowed within an uncertainty domain. Third, due to the measurement error and the modeling imprecision, the parametric uncertainties should be considered. In the chapter, we consider the system parametric uncertainties and the external disturbances, respectively and independently. A robust output feedback controller is designed to deal with the system matrix uncertain part, while a H_∞ controller is then presented to realize disturbance attenuation.

Another important part of this chapter is the optimal control problem of MJLS, which has attracted many researchers [14, 25, 33]. The majority of the studies focus on the feedback optimal regulator of jump linear system (JLS) under the assumption that the transition rate of the continuous-time Markov chain are given a prior. This assumption means that the transition among different regimes is natural or is affected by the inherent characteristic of the system itself. However, it is not the case in practice. Although the switching between different regime is random, the transition rate or probability is always affected by some external factors. For example, in a failure prone manufacturing system, an important features is that the failure rate and the frequency of preventive maintenance of the machine are relevant. In the operational regime, when the production rate is guaranteed, we can reduce the machine failure rate and improve the productivity by some maintenance policies including cleaning, lubrication, adjustment, etc. In the failure regime, some repair policy can be applied to reduce the dwell time in the failure regime. The wireless networked control system is another example. The stochastic packet loss is unavoidable in an unreliable wireless channel and the package loss rate is affected by the intensity of communication signal [22].

The relevant studies on such kind of systems are rare. Early study can be traced back to [40], where the u-dependent transition rates are considered to describe the system with switches being dependent on the value of inputs or loads. A nonlinear partial differential equation related to the optimal solution of the x- and u-dependent problem, was adopted to represent this kind of system. However, the exact solution of the nonlinear partial differential equation was not fully investigated in that work. In [24], the discrete-time jump linear quadratic (JLQ) problem was considered for JLS, where the transition probability is controlled by the choice of a finite-valued input. The optimal solution for finite and infinite time horizon were both developed. For manufacturing systems, some models were proposed in [7, 9], where the transition rate between the operational regime and the failure regime depends not only on the age of the machine, but also the frequency of maintenance. In [45], under the situation that the jumping rates are controlled, the JLQ regulator for such JLS is studied. Recently, based on the long-run average performance criterion, a gradient potential method was applied to analyze Jump Linear Quadratic Gaussian (JLQG) model in [45].

In this chapter, we still consider the JLQ problem for the continuous-time time-invariant MJLS. Differently, the switching between regimes is characterized by Markov decision processes (MDPs), i.e. the transition rate is determined by the corresponding actions and this relationship is described by the regime-dependent policy. Under the assumption that the initial policy is available, our objective is to improve the quadratic performance index given a prior, by tuning the initial policy. To this end, we employ a two-level regulating method and prove that the closed-loop Lyapunov matrix is twice differentiable with respect to the policy variable. Based on this result, we develop an algorithm to seek for a near-optimal policy by the gradient projection method, and prove the convergence of the algorithm. Furthermore, we study the near-optimal policy in special cases, and obtain some more practical results.

The Chapter is organized as follows. The system modeling of Wheeled Mobile Manipulators are given in Sect. 7.2. The output feedback controller is showed in Sect. 7.2.2. Markovian model and some definition and lemma are given in Sect. 7.2.3. Stability analysis are given in Sect. 7.2.4. The simulation studies are showed in Sect. 7.2.5. The problem of the second part of this chapter is formulated in Sect. 7.3, where some definitions and assumptions are also introduced. The main results are provided in Sect. 7.3.2, followed by two illustrative examples in Sect. 7.3.4. Finally, concluding remarks are drawn in Sect. 7.4.

7.2 Robotic Manipulator System

7.2.1 Introduction to the System

Consider a robotic manipulator with n_a degrees of freedom mounted on a two-wheeled driven mobile platform. The dynamics can be described as [29]:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) + d(t) = B(q)\tau + f, \tag{7.1}$$

where $q = [q_v^T, q_a^T]^T \in R^n$ with $q_v = [x, y, \vartheta]^T \in R^{n_v}$ denoting the generalized coordinates for the mobile platform and $q_a \in R^{n_a}$ denoting the coordinates of the robotic manipulator joints. Specifically, in this example, $n = n_v + n_a$. The symmetric positive definite inertia matrix $M(q) \in R^{n\times n} = [M_v, M_{va}; M_{av}, M_a]$, the Centripetal and Coriolis torques $C(\dot{q}, q) \in R^{n\times n} = [C_v, C_{va}; C_{av}, C_a]$, the gravitational torque vector $G(q) \in R^n = [G_v^T, G_a^T]^T$, the external disturbance $d(t) \in R^n = [d_v^T; d_a^T]^T$, the known input transformation matrix $B(q) \in R^{n\times m}$, the control inputs $\tau \in R^m$, $B(q)\tau = [\tau_v^T, \tau_a^T]^T$, and the generalized constraint forces $f \in R^n = [J_v^T \lambda_n, 0]^T$, and M_v, M_a describe the inertia matrices for the mobile platform, the links respectively, M_{va} and M_{av} are the coupling inertia matrices of the mobile platform, the links; C_v, C_a denote the Centripetal and Coriolis torques for the mobile platform, the links, respectively; C_{va}, C_{av} are the coupling Centripetal and Coriolis torques of the mobile platform, the links. G_v and G_a are the gravitational torque vectors for the mobile platform, the links, respectively; τ_v is the input vector associated with the left driven wheel and the right driven wheel, respectively; and τ_a is the control input vectors for the joints of the manipulator; d_v, d_a denote the external disturbances on the mobile platform, the links, respectively, such as a vibration tend to affect the positioning accuracy of the manipulator; $J_v \in R^{l\times n_v}$ is the kinematic constraint matrix related to nonholonomic constraints; $\lambda_n \in R^l$ is the associated Lagrangian multipliers with the generalized nonholonomic constraints. We assume that the mobile manipulator is subject to known nonholonomic constraints.

The vehicle subject to nonholonomic constraints can be expressed as $J_v\dot{q}_v = 0$. Assume that the annihilator of the co-distribution spanned by the covector fields $J_{v_1}^T(q_v), \ldots, J_{v_l}^T(q_v)$ is an $(n_v - l)$-dimensional smooth nonsingular distribution Δ on R^{n_v}. This distribution Δ is spanned by a set of $(n_v - l)$ smooth and linearly independent vector fields $H_1(q_v), \ldots, H_{n_v-l}(q_v)$, i.e. $\Delta = \text{span}\{H_1(q_v), \ldots, H_{n_v-l}(q_v)\}$, which satisfy, in local coordinates, the following relation $H^T(q_v) J_v^T(q_v) = 0$ [29], where $H(q_v) = [H_1(q_v), \ldots, H_{n_v-l}(q_v)] \in R^{n_v\times(n_v-l)}$. Note that $H^T H$ is of full rank. The nonholonomic constraintimplies the existence of vector

Table 7.1 The modes of operation

Mode	Torques					
	τ_{θ_r}	τ_{θ_l}	τ_{θ_1}	τ_{θ_2}	$\cdots$	$\tau_{\theta_{na}}$
1	Normal	Normal	Normal	Normal	$\cdots$	Normal
2	Normal	0	Normal	Normal	$\cdots$	Normal
3	Normal	0	0	Normal	$\cdots$	Normal
4	Normal	0	Normal	0	$\cdots$	Normal
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{n_a+1}	Normal	0	0	0	$\cdots$	0

$\dot{\eta} \in R^{n_v - l}$, such that $\dot{q}_v = H(q_v)\dot{\eta}$. Considering the above equation and its derivative, the dynamics of mobile manipulator can be expressed as

$$\mathscr{M}(\zeta)\ddot{\zeta} + \mathscr{C}(\zeta,\dot{\zeta})\dot{\zeta} + \mathscr{G}(\zeta) + \mathscr{D}(t) = \mathscr{U}, \tag{7.2}$$

where $\mathscr{M}(\zeta) = \begin{bmatrix} H^T M_{vH} & H^T M_{va} \\ M_{av} H & M_a \end{bmatrix}$, $\zeta = \begin{bmatrix} \eta \\ q_a \end{bmatrix}$, $\mathscr{G}(\zeta) = \begin{bmatrix} H_v^{TG} \\ G_a \end{bmatrix}$, $\mathscr{C}(\zeta,\dot{\zeta}) = \begin{bmatrix} H_v^{TM}\dot{H} + H_{vH}^{TC} & H_{va}^{TC} \\ M_{av}\dot{H} + C_{av}H & C_a \end{bmatrix}$, $\mathscr{U} = \begin{bmatrix} \tau_v^{TH} & \tau_a^T \end{bmatrix}^T$, $\mathscr{D}(t) = \begin{bmatrix} d_v^{TH} & d_a^T \end{bmatrix}^T$.

Remark 7.1 In this example, we choose $\zeta = [\theta_r, \theta_l, \theta_1, \theta_2, \cdots, \theta_{n_a}]^T$, $\eta = [\theta_r, \theta_l]^T$, and $\mathscr{U} = [\tau_r, \tau_l, \tau_1, \cdots, \tau_{n_a}]^T$.

Remark 7.2 The total degree of freedom for the reduced model of the two-wheeled driven mobile manipulator with two wheels and n_a joints is $n_q = n_a + 2$.

Now we suppose failures may appear in left wheel and each joint independently. Then 2^{n_a+1} modes of operation, can be associated to Table 7.1 depending on which torque has failed. We partition the dynamics (7.2) into two parts, the operational part and the failed part, represented by "o" and "f", respectively. Then we can rewrite the dynamics (7.2) as

$$\begin{bmatrix} M_{oo}(\zeta) & M_{of}(\zeta) \\ M_{fo}(\zeta) & M_{ff}(\zeta) \end{bmatrix} \begin{bmatrix} \ddot{\zeta}_o \\ \ddot{\zeta}_f \end{bmatrix} + \begin{bmatrix} C_{oo}(\zeta,\dot{\zeta}) & C_{of}(\zeta,\dot{\zeta}) \\ C_{fo}(\zeta,\dot{\zeta}) & C_{ff}(\zeta,\dot{\zeta}) \end{bmatrix} \begin{bmatrix} \dot{\zeta}_o \\ \dot{\zeta}_f \end{bmatrix}$$
$$+ \begin{bmatrix} G_o \\ G_f \end{bmatrix} + \begin{bmatrix} d_o(t) \\ d_f(t) \end{bmatrix} = \begin{bmatrix} B_{oo} & B_{of} \\ B_{fo} & B_{ff} \end{bmatrix} \begin{bmatrix} \tau_o \\ 0 \end{bmatrix},$$

where M_{oo},M_{of},M_{fo}, M_{ff}: the coupling inertia matrices of the operational parts and the failed parts; C_{oo},C_{of},C_{fo},C_{ff}: the Centripetal and Coriolis torque matrices of the operational parts and the failed parts; G_o,G_f: the gravitational torque vector for the operational parts and the failed parts respectively; $d_o(t)$,$d_f(t)$: the external disturbance on the operational parts and the failed parts respectively; B_{oo},B_{of},B_{fo},B_{ff}: the known full rank input transformation matrix of the operational parts and the failed parts; τ_o: the control input torque vector for the operational parts of the manipulator; τ_f: the control input torque vector for the failed parts of the manipulator satisfying $\tau_f = 0$. After some simple manipulations, we obtain

$$\bar{B}\tau_o = \bar{M}(\zeta)\ddot{\zeta}_o + \bar{H}(\zeta, \dot{\zeta}) + \bar{d}(\zeta, t), \tag{7.3}$$

where

$$\begin{aligned}
\bar{B} &= B_{oo} - M_{of} M_{ff}^{-1} B_{fo}, \\
\bar{M} &= M_{oo} - M_{of} M_{ff}^{-1} M_{fo}, \\
\bar{H}(\zeta, \dot{\zeta}) &= \bar{C}_1(\zeta, \dot{\zeta})\dot{\zeta}_o + \bar{C}_2(\zeta, \dot{\zeta})\dot{\zeta}_f + G_o - M_{of}(\zeta) M_{ff}^{-1}(\zeta) G_f(\zeta), \\
\bar{d}(\zeta, t) &= d_o(t) - M_{of}(\zeta) M_{ff}^{-1}(\zeta) d_f(t).
\end{aligned}$$

with $\bar{C}_1(\zeta, \dot{\zeta}) = C_{oo}(\zeta, \dot{\zeta}) - M_{of} M_{ff}^{-1}(\zeta) C_{fo}(\zeta, \dot{\zeta})$ and $\bar{C}_2(\zeta, \dot{\zeta}) = C_{of}(\zeta, \dot{\zeta}) - M_{of} \times M_{ff}^{-1}(\zeta) C_{ff}(\zeta, \dot{\zeta})$.

The fully operational mobile manipulator can be represented by (7.3) with $\bar{B} = B$, $\bar{M}(\zeta) = M(\zeta)$, $\bar{H}(\zeta, \dot{\zeta}) = C(\zeta, \dot{\zeta})\dot{\zeta} + G$, $\bar{d}(\zeta, t) = d(t)$. Then, by linearizing the dynamics (7.3) around an operation point with position q_0 and velocity $\dot{q}_0$, we have the following linear system

$$\begin{cases} \dot{x} = \bar{A}(\zeta_0, \dot{\zeta}_0)x + \bar{B}(\zeta_0)u + \bar{W}(\zeta_0)w \\ z = \bar{C}x + \bar{D}u \end{cases} \tag{7.4}$$

where

$$\begin{aligned}
\bar{A}(\zeta_0, \dot{\zeta}_0) &= \left.\begin{bmatrix} 0 & I \\ -\frac{\partial}{\partial \zeta^T}(\bar{M}^{-1}(\zeta)\bar{H}(\zeta, \dot{\zeta})) & -\bar{M}^{-1}(\zeta)\frac{\partial}{\partial \dot{\zeta}^T}(\bar{H}(\zeta, \dot{\zeta})) \end{bmatrix}\right|_{\zeta_0, \dot{\zeta}_0}, \\
\bar{B}(\zeta_0) &= \left.\begin{bmatrix} 0 \\ \bar{M}^{-1}(\zeta)\bar{B} \end{bmatrix}\right|_{\zeta_0} \quad \bar{W}(\zeta_0) = \left.\begin{bmatrix} 0 \\ \bar{M}^{-1}(\zeta) \end{bmatrix}\right|_{\zeta_0}.
\end{aligned}$$

s

and $x = [\zeta^d - \zeta, \dot{\zeta}^d - \dot{\zeta}]^T$ represents the state tracking error, z, $u = \tau_o$, $w = d(t)$ represent the controlled output, the control input and exogenous disturbance, respectively, and $\bar{C}$ and $\bar{D}$ are constant matrices defined by the designer and are used to adjust the Markovian controllers.

7.2.2 Output Feedback Controller Based on High-Gain Observer

Since it may be difficult to measure the velocity signal, only the position signal $\zeta^d - \zeta$ is measurable, we need to estimate x to implement the feedback control. Therefore, a high-gain observer is employed to estimate the states of the system.

Lemma 7.1 *Suppose the function $y(t)$ and its first n derivatives are bounded. Consider the following linear system*

$$\begin{aligned} &\epsilon\dot{\xi}_1 = \xi_2,\ \epsilon\dot{\xi}_2 = \xi_3,\ \ldots,\ \epsilon\dot{\xi}_{n-1} = \xi_n, \\ &\epsilon\dot{\xi}_n = -b_1\xi_n - b_2\xi_{n-1} - \cdots - b_{n-1}\xi_2 - \xi_1 + y(t), \end{aligned} \tag{7.5}$$

where the parameters b_1 to b_{n-1} are chosen so that the polynomial $s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + 1$ is Hurwitz. Then, there exist positive constants $h_k,\ k = 2, 3, \ldots, n$ and t^* such that for all $t > t^*$ we have

$$\frac{\xi_{k+1}}{\epsilon^k} - y^{(k)} = -\epsilon\psi^{(k+1)}, \quad k = 1, \ldots, n-1 \tag{7.6}$$

$$|\frac{\xi_{k+1}}{\epsilon^k} - y^{(k)}| \le \epsilon h_{k+1}, \quad k = 1, \ldots, n-1 \tag{7.7}$$

where ϵ is any small positive constant, $\psi = \xi_n + b_1\xi_{n-1} + \ldots + b_{n-1}\xi_1$ and $|\psi^{(k)}| \le h_k$. $\psi^{(k)}$ denotes the kth derivative of ψ.

Proof The proof can be found in [1].

Let the measured output $y(t) = [y_1^T(t), y_2^T(t), \ldots, y_{n_q}^T(t)]^T \in \mathbb{R}^{n_q}$ be the position tracking error signal $\zeta^d - \zeta$ measured in the n_q-link manipulator system (7.4). Applying observer (7.5), we define the following variables ($j = 1, 2, \ldots, n_q$):

$$\begin{aligned} &\epsilon\dot{\xi}_{j1} = \xi_{j2}, && \epsilon\dot{\xi}_{j2} = -b_j\xi_{j2} - \xi_{j1} + y_j(t), \\ &\xi_j(t) = [\xi_{j1}^T(t), \xi_{j2}^T(t)]^T, && \hat{x}_j(t) = [y_j^T(t), \frac{\xi_{j2}^T(t)}{\epsilon}]^T. \end{aligned}$$

Transform the above equations into matrix form, we get

$$\begin{cases} \dot{\xi}(t) = M\xi(t) + Ny(t) \\ \hat{x}(t) = M_p\xi(t) + N_p y(t) \end{cases} \tag{7.8}$$

where

$$\xi(t)=[\xi_1^T(t), \xi_2^T(t), \ldots, \xi_{n_q}^T(t)]^T, \quad \hat{x}(t)=[\hat{x}_1^T(t), \hat{x}_2^T(t), \ldots, \hat{x}_{n_q}^T(t)]^T,$$
$$M = diag\{\mathfrak{M}_1, \ldots, \mathfrak{M}_{n_q}\}, \quad \mathfrak{M}_j = [0, \frac{1}{\epsilon}; -1/\epsilon, -\frac{b_j}{\epsilon}];$$
$$N = diag\{\mathfrak{N}_1, \ldots, \mathfrak{N}_{n_q}\}, \quad \mathfrak{N}_j = [0, 1/\epsilon]^T;$$
$$M_p = diag\{\mathfrak{M}_{p1}, \ldots, \mathfrak{M}_{pn_q}\}, \quad \mathfrak{M}_{pj} = [0, 0; 0, 1/\epsilon];$$
$$N_p = diag\{\mathfrak{N}_{p1}, \ldots, \mathfrak{N}_{pn_q}\}, \quad \mathfrak{N}_{pj} = [1, 0]^T,$$

with the chosen parameters b_j so that the polynomial $s^2 + b_j s + 1$ is Hurwitz, for $j = 1, 2, \ldots, n_q$.

The output feedback controller on observer (7.8) is given by

$$u(t) = K\hat{x}(t), \tag{7.9}$$

where K is the controller gain to be designed.

Remark 7.3 The output feedback controller proposed here is easy to implement because it is simply a state feedback design with a linear high-gain observer without a priori knowledge of the nonlinear systems. Unlike other exact linearization approach, it is not necessary to search for a nonlinear transformation and an explicit control function. Moreover, the high-gain observer has certain disturbance rejection and linearization properties.

7.2.3 Markovian Model and Problem Statement

The Markovian model developed in this section not only contains the transition among the operation points in system (7.4), but also describes the probability of a fault occurrence. For a mobile manipulator with 2 wheels and n_a joints, we have totally 2^{n_a+1} possible configurations as discussed in Sect. 7.2.1. Now we proceed to consider the linearization configurations.

Note that although the transitions among the plant linearization points are not a genuine stochastic event in contrast with the moment of a fault occurrence, the Markovian techniques can be applied in this case since the Markovian transition rate is related with the expected mean time the system is supposed to lie in each state of the Markovian chain. We may consider the workspace of each joint with a positioning domain which range from ζ_{p_1} to ζ_{p_2}, with the velocities set to zero, and divide the workspace into n_p sectors. For each range of $(\zeta_{p_2} - \zeta_{p_1})/n_p$ of each joint, it is defined as a linearization point for the manipulator. For each linearization point, there exist 2 sets of matrices $\bar{A}(\zeta_0, \dot{\zeta}_0)$, $\bar{B}(\zeta_0)$, $\bar{W}(\zeta_0)$, $\bar{C}$, $\bar{D}$ corresponding to all the 2^{n_a+1} configurations. Hence, the Markovian modes are the manipulator dynamic model linearized properly according to (7.4) in these linearization points for all configurations. The choice of these sectors and the number of the sectors n_p need to be firstly decided in order to guarantee the effectiveness of the Markovian jump model.

Then, the number of all the possible combinations of positioning of the 2 wheels and n_a joints $\theta_1, \theta_2, \ldots, \theta_{n_a}$ may be computed as $2^{n_a+1}n_p^{n_a+2}$, i.e. $n+2$ linearization points with $2^{n_a+1}n_p^{n_a+2}$ modes are found, which means a $2^{n_a+1}n_p^{n_a+2} \times 2^{n_a+1}n_p^{n_a+2}$ transition rate matrix Π is needed. To distinguish the operation point level and the fault occurrence level more clearly, we may partition Π into 2^{n_a+1} blocks with each block be an $n_p^{n_a+2} \times n_p^{n_a+2}$ matrix. An illustrative example will be shown in Sect. 7.2.5.

Remark 7.4 The dimensions of the matrix sets $\{\bar{A}(\zeta_0, \dot{\zeta}_0), \bar{B}(\zeta_0), \bar{W}(\zeta_0), \bar{C}, \bar{D}\}$ may be different among all the configurations. While applying Markovian method, lines and columns of zeros may be introduced to make sure that the system matrix sets of all modes have the same dimension.

Remark 7.5 Note that all the elements of Π are selected empirically, which means they cannot be precisely known as a priori. In order to tackle the estimation error, we consider a more realistic way where the nominal estimated value of the transition rate matrix is measured in practice and error bounds are given. This problem will be described in detail in Sect. 7.2.4.

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (increasing and right continuous and $\mathscr{F}_0$ contains all P-null sets). We denote by $\mathbb{L}_2[0, \infty)$ the Hilbert space formed by the stochastic process $z = \{z(t); t \geq 0\}$ such that, for each $t \geq 0$, $z(t)$ is a second order real valued random vector, $\mathscr{F}_t$-measurable and $||z||_2^2 \triangleq \int_0^\infty E\{||z(t)||^2\}dt < \infty$. Consider the following hybrid system:

$$\begin{cases} \dot{x}(t) = A(r(t))x(t) + B(r(t))u(t) + W(r(t))w(t) \\ z(t) = C(r(t))x(t) + D(r(t))u(t) \\ y(t) = E(r(t))x(t) \\ x(0) = x_0, \quad r(0) = r_0 \end{cases} \tag{7.10}$$

where $x(\cdot) \in \mathbb{R}^{n_q}$, $u(\cdot) \in \mathbb{R}^{n_u}$, $z(\cdot) \in \mathbb{R}^{n_z}$, $y(\cdot) \in \mathbb{R}^{n_y}$ are, respectively, the state trajectory, the input, the controlled output, and the measured output for the system (7.10). $w(\cdot) \in \mathbb{R}^{n_w}$ is the exogenous disturbance signal that belongs to $\mathbb{L}_2[0, \infty)$. $A(\cdot), B(\cdot), W(\cdot), C(\cdot), D(\cdot), E(\cdot)$ are real constant matrices with appropriate dimensions. These matrices are given by the system (7.4). $r(\cdot)$ is a homogeneous Markov process taking value in a finite state space $\mathscr{S} = \{1, 2, \ldots, N\}$ with generator Π showed in (1.3). The probability initial distribution of the Markov process is given by $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ in such a way that $P(r_0 = i) = \mu_i$. As mentioned in Sect. 7.2.1, both system matrices $A(r(t))$, $B(r(t))$ and the mode transition rate matrix Π are not precisely known as a priori. The following Assumptions are in order.

Assumption 7.1 Divide the parameter matrices $A(r(t))$, $B(r(t))$ into a nominal part and a perturbed part

$$A(r(t)) = \bar{A}(r(t)) + \Delta A(r(t), t),\ B(r(t)) = \bar{B}(r(t)) + \Delta B(r(t), t), \quad (7.11)$$

where the uncertain parameters are assumed to be in following forms:

$$\begin{aligned} \Delta A(r(t), t) &= H_a(r(t)F(r(t), t)L(r(t)), \\ \Delta B(r(t), t) &= H_b(r(t)F(r(t), t)L(r(t)), \end{aligned} \quad (7.12)$$

where $H_a(r(t))$, $H_b(r(t))$, $L(r(t))$ are known constant real matrices of appropriate dimensions, while $F(r(t), t)$ denotes the uncertainties in the system matrices satisfying $F^T(r(t), t)F(r(t), t) \leq I, \forall r(t) \in \mathscr{S}$.

Assumption 7.2 The mode transition rate matrix belongs to the following admissible uncertainty domain:

$$\mathscr{D}_\pi \triangleq \{\bar{\Pi} + \Delta\Pi : |\Delta\pi_{ij}| \leq \varepsilon_{ij},\ \varepsilon_{ij} \geq 0, \forall i, j \in \mathscr{S},\ j \neq i\}, \quad (7.13)$$

where $\bar{\Pi} \triangleq (\bar{\pi}_{ij})$ is a known constant transition rate matrix, while $\Delta\Pi \triangleq (\Delta\pi_{ij})$ denotes the uncertainty. For all $i, j \in \mathscr{S}$, $j \neq i$, $\bar{\pi}_{ij} (\geq 0)$ denotes the estimated value of π_{ij}, and the error between them is referred as to $\Delta\pi_{ij}$ which can take any value in $[-\varepsilon_{ij}, \varepsilon_{ij}]$; For all $i, j \in \mathscr{S}$, $\bar{\pi}_{ii} = -\sum_{j\in\mathscr{S}, j\neq i} \bar{\pi}_{ij}$ and $\Delta\pi_{ii} = -\sum_{j\in\mathscr{S}, j\neq i} \Delta\pi_{ij}$.

Remark 7.6 The estimation error bound ε_{ij} could be determined empirically from an admissible portion of the nominal value π_{ij} which is the estimated value of the mode transition rate after lots of statistics in practice, for example, 10% of π_{ij}.

A dynamic output feedback controller based on the high-gain observer is adopted to solve the problem of fault-tolerant manipulator control described in Sect. 7.2.1. According to (7.8) and (7.9), the linear mode-dependent output control law is given by:

$$\begin{cases} \dot{\xi}(t) = M\xi(t) + Ny(t) \\ u(t) = K(r(t))M_p\xi(t) + K(r(t))N_p y(t) \end{cases} \quad (7.14)$$

It is possible to incorporate both systems (7.10) and (7.14), into a closed-loop system, with the augmented state variable $\bar{\zeta}(t) = [x^T(t), \xi^T(t)]^T \in \mathbb{R}^{2n}$ for any $t \geq 0$. The state and output equations for this $2n$-dimensional system may be written as:

$$\begin{cases} \dot{\zeta}(t) = A_a(r(t))\bar{\zeta}(t) + W_a(r(t))w(t) \\ z(t) = C_a(r(t))\bar{\zeta}(t) \end{cases} \quad (7.15)$$

where

$$
\begin{aligned}
A_a(r(t)) &= \begin{bmatrix} A_{a11} & B(r(t))K(r(t))M_p \\ NE(r(t)) & M \end{bmatrix}, \\
A_{a11} &= A(r(t)) + B(r(t))K(r(t))N_pE(r(t)), \\
C_a(r(t)) &= \begin{bmatrix} C(r(t)) + D(r(t))K(r(t))N_pE(r(t)) \\ D(r(t))K(r(t))M_p \end{bmatrix}^T, \\
W_a(r(t)) &= \begin{bmatrix} W(r(t)) \\ 0 \end{bmatrix}.
\end{aligned}
$$

For simplicity, in the sequel, let M_i denote the corresponding matrix, $M(r(t))$, for each $i \in \mathscr{S}$. The weak infinitesimal generator, acting on functional $V : \mathbb{C}(\mathbb{R}^n \times \mathscr{S} \times \mathbb{R}_+) \to \mathbb{R}$, is defined by $\mathfrak{L}V(x(t), i, t) = \lim_{\Delta \to 0_+} \frac{1}{\Delta} \times \{E[V(x(t + \Delta), r(t + \Delta), t + \Delta)|x(t), r(t) = i] - V(x(t), i, t)\}$. For further references on the associated operator of the hybrid system (7.10), we suggest the reader to see related work in [27, 41].

Definition 7.1 [18] The system (7.10) with $w \equiv 0$ is said to be stochastic stable (SS) if $\int_0^{+\infty} E\{||x(t; x_0, r_0)||^2\}dt < +\infty$, for any finite initial condition $x_0 \in \mathbb{R}^n$ and any initial distribution for $r_0 \in \mathscr{S}$.

Lemma 7.2 *[43] Given matrices $Q = Q^T$, H, E and $R = R^T > 0$ of appropriate dimensions, $Q + HFE + E^T F^T H^T < 0$, for all F satisfying $F^T F \leq R$, if and only if there exists some real number $\lambda \in \mathbb{R}^+$ such that $Q + \lambda HH^T + \lambda^{-1} E^T RE < 0$.*

Lemma 7.3 *[34] Given matrices D, F and H of appropriate dimensions with F satisfying $F^T F \leq I$. Then for any scalar $\varepsilon > 0$ and vectors x, y, $2x^T DFHy \leq \frac{1}{\varepsilon} x^T DD^T x + \varepsilon y^T H^T Hy$.*

7.2.4 Stability Analysis

7.2.4.1 Robust Stochastic Stability

Firstly, we consider the robust stochastic stability for the system (7.10) with uncertainty domain (7.11) when $w \equiv 0$.

Theorem 7.1 *The Markovian jump system (7.10) ($w(t) \equiv 0$) with uncertainty domain (7.11) and dynamic output feedback control law (7.14) is robustly stochastic stable if, for any $i, j \in \mathscr{S}$, $i \neq j$, there exist positive-definite matrices $X_i, Y_i \in \mathbb{R}^{n \times n}$, $K_i \in \mathbb{R}^{n_u \times n}$ and positive real numbers λ_{ij} such that the LMIs (7.16) holds*

$$\begin{bmatrix} \Phi_{11i} & \Phi_{12i} & \Psi_{1i} & 0 & \Gamma_{1i} & \Gamma_{2i} & \Gamma_{3i} & \Gamma_{4i} & \Gamma_{5i} \\ * & \Phi_{22i} & 0 & \Psi_{2i} & 0 & M_p^T K_i^T & M_p^T K_i^T L_i^T & 0 & 0 \\ * & * & -\Lambda_i & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\Lambda_i & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\epsilon_i I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{7.16}$$

where $\Phi_{11i} = \bar{A}_i^T X_i + X_i \bar{A}_i + \sum\limits_{j \in \mathscr{S}} \bar{\pi}_{ij} X_j + \sum_{j \in \mathscr{S}, j \neq i} \frac{\lambda_{ij}}{4} \varepsilon_{ij}^2 I_n + \epsilon_i L_i^T L_i$, $\Phi_{12i} = E_i^T N^T Y_i$, $\Gamma_{1i} = X_i H_{ai}$, $\Gamma_{2i} = E_i^T N_p^T K_i^T$, $\Gamma_{3i} = E_i^T N_p^T K_i^T L_i^T$, $\Gamma_{4i} = X_i \bar{B}_i$, $\Gamma_{5i} = X_i H_{bi}$, $\Phi_{22i} = M^T Y_i + Y_i M + \sum\limits_{j \in \mathscr{S}} \bar{\pi}_{ij} Y_j + \sum_{j \in \mathscr{S}, j \neq i} \frac{\lambda_{ij}}{4} \varepsilon_{ij}^2 I_n$, $\Psi_{1i} = [X_i - X_1, X_i - X_2, \ldots, X_i - X_{i-1}, X_i - X_{i+1}, \ldots, X_i - X_N]$, $\Psi_{2i} = [Y_i - Y_1, Y_i - Y_2, \ldots, Y_i - Y_{i-1}, Y_i - Y_{i+1}, \ldots, Y_i - Y_N]$, $\Lambda_i = diag\{\lambda_{i1} I_n, \lambda_{i2} I_n, \ldots, \lambda_{i(i-1)} I_n, \lambda_{i(i+1)} I_n, \ldots, \lambda_{iN} I_n\}$.

Proof We consider the equivalent closed-loop Markovian jump linear system (7.15) without disturbance (i.e. $w(t) \equiv 0$). For each $r(t) = i,\ i \in \mathscr{S}$, we define a positive-definite matrix $P_i \in \mathbb{R}^{2n \times 2n}$ by $P_i = \mathrm{diag}[X_i, Y_i]$. Then we construct a stochastic Lyapunov functional candidate as $V(\bar{\zeta}(t), i, t) = \bar{\zeta}^T(t) P_i \bar{\zeta}(t)$ Applying the Markovian infinitesimal operator, we have

$$\mathfrak{L}V(\bar{\zeta}(t), i, t) = \bar{\zeta}^T(t)(A_{ai}^T P_i + P_i A_{ai} + \sum_{j \in \mathscr{S}} \pi_{ij} P_j)\bar{\zeta}(t). \tag{7.17}$$

With (7.11), (7.12) and (7.13), we have

$$\begin{aligned} \mathfrak{L}V(\bar{\zeta}(t), i, t) = {} & \bar{\zeta}^T(t)\Bigg(\Omega_i + \begin{bmatrix} L_i^T F_i^T \\ 0 \end{bmatrix} \begin{bmatrix} H_{ai}^T X_i & 0 \end{bmatrix} + \begin{bmatrix} X_i H_{ai} \\ 0 \end{bmatrix} \begin{bmatrix} F_i L_i & 0 \end{bmatrix} \\ & + \sum_{j \in \mathscr{S}} \bar{\pi}_{ij} P_j + \begin{bmatrix} E_i^T N_p^T K_i^T & E_i^T N_p^T K_i^T L_i^T \\ M_p^T K_i^T & M_p^T K_i^T L_i^T \end{bmatrix} \begin{bmatrix} I_{n_u} & 0 \\ 0 & F_i^T \end{bmatrix} \begin{bmatrix} \bar{B}_i^T X_i & 0 \\ H_{bi}^T X_i & 0 \end{bmatrix} \\ & + \begin{bmatrix} X_i \bar{B}_i & X_i H_{bi} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{n_u} & 0 \\ 0 & F_i \end{bmatrix} \begin{bmatrix} K_i N_p E_i & K_i M_p \\ L_i K_i N_p E_i & L_i K_i M_p \end{bmatrix} \\ & + \sum_{j \in \mathscr{S}, j \neq i} [\frac{1}{2} \Delta\pi_{ij}(P_j - P_i) + \frac{1}{2} \Delta\pi_{ij}(P_j - P_i)]\Bigg)\bar{\zeta}(t) \\ \triangleq {} & \bar{\zeta}^T(t) \Xi_i \bar{\zeta}(t), \end{aligned} \tag{7.18}$$

where $\Omega_i = \begin{bmatrix} \bar{A}_i^T X_i + X_i \bar{A}_i & E_i^T N^T Y_i \\ Y_i N E_i & M^T Y_i + Y_i M \end{bmatrix}$.

From Lemma 7.2, we know that $\Xi_i < 0$ if and only if there exist real numbers $\rho_i \in \mathbb{R}^+$ such that

$$\begin{aligned}
&\Omega_i + \begin{bmatrix} L_i^T F_i^T \\ 0 \end{bmatrix} \begin{bmatrix} H_{ai}^T X_i \; 0 \end{bmatrix} + \begin{bmatrix} X_i H_{ai} \\ 0 \end{bmatrix} \begin{bmatrix} F_i L_i \; 0 \end{bmatrix} + \sum_{j \in \mathscr{S}} \bar{\pi}_{ij} P_j \\
&+ \rho_i^{-1} \begin{bmatrix} E_i^T N_p^T K_i^T & E_i^T N_p^T K_i^T L_i^T \\ M_p^T K_i^T & M_p^T K_i^T L_i^T \end{bmatrix} \begin{bmatrix} K_i N_p E_i & K_i M_p \\ L_i K_i N_p E_i & L_i K_i M_p \end{bmatrix} \\
&+ \rho_i \begin{bmatrix} X_i \bar{B}_i & X_i H_{bi} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{B}_i^T X_i & 0 \\ H_{bi}^T X_i & 0 \end{bmatrix} + \sum_{j \in \mathscr{S}, j \neq i} [\frac{1}{2} \Delta\pi_{ij} (P_j - P_i) \\
&+ \frac{1}{2} \Delta\pi_{ij} (P_j - P_i)] < 0
\end{aligned}$$

holds for each $i \in \mathscr{S}$.

Now multiply the above equation by ρ_i on both sides and replace ρX_i, ρY_i with X_i, Y_i. Then using Lemma 7.2 again, it is deduced that the above inequality holds if and only if there exist real numbers $\epsilon_i, \lambda_{ij} \in \mathbb{R}^+$ such that

$$\begin{aligned}
&\Omega_i + \epsilon_i \begin{bmatrix} L_i^T \\ 0 \end{bmatrix} \begin{bmatrix} L_i & 0 \end{bmatrix} + \epsilon_i^{-1} \begin{bmatrix} X_i H_{ai} \\ 0 \end{bmatrix} \begin{bmatrix} H_{ai}^T X_i & 0 \end{bmatrix} + \sum_{j \in \mathscr{S}} \bar{\pi}_{ij} P_j \\
&+ \begin{bmatrix} E_i^T N_p^T K_i^T & E_i^T N_p^T K_i^T L_i^T \\ M_p^T K_i^T & M_p^T K_i^T L_i^T \end{bmatrix} \begin{bmatrix} K_i N_p E_i & K_i M_p \\ L_i K_i N_p E_i & L_i K_i M_p \end{bmatrix} \\
&+ \begin{bmatrix} X_i \bar{B}_i & X_i H_{bi} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{B}_i^T X_i & 0 \\ H_{bi}^T X_i & 0 \end{bmatrix} + \sum_{j \in \mathscr{S}, j \neq i} [\frac{\lambda_{ij}}{4} \varepsilon_{ij}^2 I_{2n} + \frac{1}{\lambda_{ij}} (P_j - P_i)^2] < 0
\end{aligned}$$

holds for each $i, j \in \mathscr{S}$, which is equivalent to (7.16) in view of Schur complement equivalence.

Hence, the LMIs (7.16) guarantee that $\Xi_i < 0$. Then we have $\mathfrak{L} V(\bar{\zeta}(t), i, t) = \bar{\zeta}^T(t) \Xi_i \bar{\zeta}(t) < 0$. We choose $\alpha = \max_{i \in \mathscr{S}} \lambda_{\max} \Xi_i$. Obviously, $\alpha < 0$.

It follows that

$$\mathfrak{L} V(\bar{\zeta}(t), i, t) \leq \alpha ||\bar{\zeta}(t)||^2 \leq \alpha ||x(t)||^2.$$

Using Dynkin Formula [12], we have

$$E\{V(x(t), i, t)\} - V(x_0, r_0, 0) \leq \alpha E\{\int_0^t ||x(s)||^2 ds\},$$

which, together with $E\{V(x(t), i, t)\} \geq 0$, implies

$$E\{\int_0^t ||x(s)||^2 ds\} \leq \alpha^{-1} \big\{E\{V(x(t), i, t)\} - V(x_0, r_0, 0)\big\}.$$

Letting $t \to \infty$ and noting that $(-\alpha)^{-1}V(x_0, r_0, 0) < +\infty$, we know the system (7.10) achieves robustly stochastic stable according to Definition 7.1. This completes the proof.

7.2.4.2 Robust H_∞ Disturbance Attenuation

Consider the H_∞ performance function as

$$J_T = E\{\int_0^T [z^T(t)z(t) - \gamma^2 w^T(t)w(t)]dt\}, \tag{7.19}$$

for $T > 0$. The following theorem gives the result.

Theorem 7.2 *The Markovian jump system (7.10) with uncertainty domain (7.11) and dynamic output feedback control law (7.14) is robustly stochastic stable with γ-disturbance H_∞ attenuation if, for any $i, j \in \mathscr{S}$, $i \neq j$, there exist positive-definite matrices $X_i, Y_i \in \mathbb{R}^{n\times n}$, $K_i \in \mathbb{R}^{n_u\times n}$ and positive real numbers ϵ_i, λ_{ij} such that the LMIs (7.20) holds*

$$\begin{bmatrix} \Phi_{11i} & \Phi_{12i} & \Psi_{1i} & 0 & \Gamma_{1i} & \Gamma_{2i} & \Gamma_{3i} & \Gamma_{4i} & \Gamma_{5i} & \Gamma_{6i} & \Gamma_{7i} \\ * & \Phi_{22i} & 0 & \Psi_{2i} & 0 & M_p^T K_i^T & M_p^T K_i^T L_i^T & 0 & 0 & M_p^T K_i^T D_i^T & 0 \\ * & * & -\Lambda_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\Lambda_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\epsilon_i I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\rho_i I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\rho_i I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\rho_i^{-1} I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -\rho_i^{-1} I & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & * & * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{7.20}$$

for some given scalars $\rho_i > 0$*, where*

$$\begin{aligned}
&\Phi_{11i}=\bar{A}_i^T X_i+X_i\bar{A}_i+\sum_{j\in\mathscr{S}}\bar{\pi}_{ij}X_j+\sum_{j\in\mathscr{S},j\neq i}\frac{\lambda_{ij}}{4}\varepsilon_{ij}^2 I_n+\epsilon_i L_i^T L_i,\\
&\Phi_{12i}=E_i^T N^T Y_i,\ \Phi_{22i}=M^T Y_i+Y_i M+\sum_{j\in\mathscr{S}}\bar{\pi}_{ij}Y_j+\sum_{j\in\mathscr{S},j\neq i}\frac{\lambda_{ij}}{4}\varepsilon_{ij}^2 I_n,\\
&\Gamma_{1i}=X_i H_{ai},\ \Gamma_{2i}=E_i^T N_p^T K_i^T,\ \Gamma_{3i}=E_i^T N_p^T K_i^T L_i^T,\ \Gamma_{4i}=X_i\bar{B}_i,\\
&\Gamma_{5i}=X_i H_{bi},\ \Gamma_{6i}=C_i^T+E_i^T N_p^T K_i^T D_i^T,\ \Gamma_{7i}=X_i W_i,\\
&\Psi_{1i}=[X_i-X_1, X_i-X_2, \ldots, X_i-X_{i-1}, X_i-X_{i+1},\ldots, X_i-X_N],\\
&\Psi_{2i}=[Y_i-Y_1, Y_i-Y_2, \ldots, Y_i-Y_{i-1}, Y_i-Y_{i+1}, \ldots, Y_i-Y_N],\\
&\Lambda_i=diag\{\lambda_{i1}I_n, \lambda_{i2}I_n, \ldots, \lambda_{i(i-1)}I_n, \lambda_{i(i+1)}I_n, \ldots, \lambda_{iN}I_n\}.
\end{aligned}$$

Proof Consider the equivalent closed-loop Markovian jump linear system (7.15), it can be easy to obtain the condition (7.16) from (7.20). Hence, the closed-loop system (7.15) is robustly stochastic stable. Similar as Theorem 7.1, for each $r(t)=i,\ i\in\mathscr{S}$, we define a positive-definite matrix $P_i\in\mathbb{R}^{2n\times 2n}$ by $P_i=\text{diag}[X_i, Y_i]$, and construct a stochastic Lyapunov functional candidate as $V(\bar{\zeta}(t), i, t)=\bar{\zeta}^T(t)P_i\bar{\zeta}(t)$. Applying the Markovian infinitesimal operator, we have

$$\mathfrak{L}V(\bar{\zeta}(t), i, t)=\eta^T(t)\begin{bmatrix}\Xi_i & P_i W_{ai}\\ W_{ai}^T P_i & 0\end{bmatrix}\eta(t), \tag{7.21}$$

where $\eta(t)=\left[\bar{\zeta}^T(t)\ w^T(t)\right]^T$, Ξ_i is defined in (7.18).

Then by using Lemma 7.3, it follows that

$$\begin{aligned}
\Xi_i \le\ & \Omega_i+\begin{bmatrix}L_i^T F_i^T\\ 0\end{bmatrix}\begin{bmatrix}H_{ai}^T & \\ X_i & 0\end{bmatrix}+\begin{bmatrix}X_i H_{ai}\\ 0\end{bmatrix}\begin{bmatrix}F_i L_i & 0\end{bmatrix}\\
&+\rho_i^{-1}\begin{bmatrix}E_i^T N_p^T K_i^T & E_i^T N_p^T K_i^T L_i^T\\ M_p^T K_i^T & M_p^T K_i^T L_i^T\end{bmatrix}\begin{bmatrix}K_i N_p E_i & K_i M_p\\ L_i K_i N_p E_i & L_i K_i M_p\end{bmatrix}\\
&+\sum_{j\in\mathscr{S},j\neq i}[\frac{1}{2}\Delta\pi_{ij}(P_j-P_i)+\frac{1}{2}\Delta\pi_{ij}(P_j-P_i)]\\
&+\rho_i\begin{bmatrix}X_i\bar{B}_i & X_i H_{bi}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\bar{B}_i^T X_i & 0\\ H_{bi}^T X_i & 0\end{bmatrix}+\sum_{j\in\mathscr{S}}\bar{\pi}_{ij}P_j\\
\triangleq\ & \hat{\Xi}_i,
\end{aligned}$$

with ρ_i any real positive number for each $i\in\mathscr{S}$. Using Dynkin's formula again, we have $E\{V(x(T), i, T)\}-V(x_0, r_0, 0)=E\{\int_0^T \mathfrak{L}V(x(s), i, s)ds\}$. Observing the zero initial condition $V(x_0, r_0, 0)=0$ and considering the performance function, for any $w(t)\in\mathbb{L}_2[0,\infty)$, we have

$$\begin{aligned} & J_T \\ &= E\{\int_0^T [z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \mathfrak{L}V(x(t), i, t)]dt\} - E\{V(x(T), i, T)\} \\ &\leq E\{\int_0^T [z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \mathfrak{L}V(x(t), i, t)]dt\}. \end{aligned}$$

Taking (7.21) into the above inequality gives

$$\begin{aligned} J_T \leq E\Bigg\{ \int_0^T \eta^T(t) * \Bigg(& \begin{bmatrix} \hat{\Xi}_i & P_i W_{ai} \\ W_{ai}^T P_i & -\gamma^2 I_{n_w} \end{bmatrix} \\ & + \begin{bmatrix} C_i^T + E_i^T N_p^T K_i^T D_i^T \\ M_p^T D_i^T D_i^T \\ 0 \end{bmatrix} \begin{bmatrix} C_i + D_i K_i N_p E_i \\ D_i K_i M_p \\ 0 \end{bmatrix}^T \Bigg) \eta(t) dt \Bigg\}. \end{aligned}$$

From Lemma 7.2 and Schur complement, we have $J_T < 0$ if and only if there exist real numbers $\epsilon_i, \lambda_{ij} \in \mathbb{R}^+$ such that

$$\begin{aligned} & \Omega_i + \epsilon_i \begin{bmatrix} L_i^T \\ 0 \end{bmatrix} \begin{bmatrix} L_i & 0 \end{bmatrix} + \epsilon_i^{-1} \begin{bmatrix} X_i H_{ai} \\ 0 \end{bmatrix} \begin{bmatrix} H_{ai}^T X_i & 0 \end{bmatrix} \\ & + \rho_i^{-1} \begin{bmatrix} E_i^T N_p^T K_i^T & E_i^T N_p^T K_i^T L_i^T \\ M_p^T K_i^T & M_p^T K_i^T L_i^T \end{bmatrix} \begin{bmatrix} K_i N_p E_i & K_i M_p \\ L_i K_i N_p E_i & L_i K_i M_p \end{bmatrix} \\ & + \rho_i \begin{bmatrix} X_i \bar{B}_i & X_i H_{bi} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{B}_i^T X_i & 0 \\ H_{bi}^T X_i & 0 \end{bmatrix} + \gamma^{-2} W_{ai}^T P_i P_i W_{ai} \\ & + \sum_{j \in \mathscr{S}} \bar{\pi}_{ij} P_j + \sum_{j \in \mathscr{S}, j \neq i} [\frac{\lambda_{ij}}{4} \varepsilon_{ij}^2 I_{2n} + \frac{1}{\lambda_{ij}} (P_j - P_i)^2] \\ & + \begin{bmatrix} C_i + D_i K_i N_p E_i \\ D_i K_i M_p \end{bmatrix} \begin{bmatrix} C_i + D_i K_i N_p E_i \\ D_i K_i M_p \end{bmatrix}^T < 0 \end{aligned}$$

holds for any given ρ_i and each $i, j \in \mathscr{S}$. Applying Schur complement and letting $T \to \infty$, it is verified that (7.20) guarantees $J_\infty < 0$ for any $w(t) \in \mathscr{L}_2[0, \infty)$, which in turn guarantees γ-disturbance H_∞ attenuation of the closed-loop system (7.15) from $w(t)$ to $z(t)$.

Remark 7.7 Theorem 7.2 presents a sufficient condition for the solvability of the robust H_∞ control problem via output feedback controllers based on a high-gain observer. It can be seen that the condition in (7.20) is not an LMI with respect to the parameter ϵ_i since ϵ_i appears in (7.20) in a nonlinear fashion. Note that ϵ_i can be any scalar in view of Lemma 7.3. Hence, as we have applied in Theorem 7.2, an easy way to design an output feedback controller is to fix the parameter ϵ_i to solve a strict LMI in X_i, Y_i and K_i, which defines a convex solution set; such an approach was also adopted in [16, 23].

Remark 7.8 Although the method proposed in Theorem 7.2 is an extension of Theorem 7.1, it may cause some conservativeness compared with Theorem 7.1 since the parameter ρ_i has been already given. In the case when ρ_i is not fixed, it can be shown that (7.20) is equivalent to a bilinear matrix inequality (BMI). Therefore, if one can afford more computational efforts, better results will be obtained by solving this BMI directly, which can be implemented by resorting to some effective algorithms, such as the Lagrangian dual global optimization algorithm and the branch-and-cut algorithm proposed in the works by [20, 39].

7.2.5 Numerical Simulation

The following variables have been chosen to describe the wheeled mobile manipulator (see also Fig. 7.1): τ_l, τ_r: the torques of two wheels respectively; τ_1: the torques of the under-actuated joint, that is, $\tau_1 = 0$; θ_l, θ_r: the rotation angle of the left wheel and the right wheel of the mobile platform respectively; v: the forward velocity of the mobile platform; θ: the direction angle of the mobile platform; ω: the rotation velocity of the mobile platform, and $\omega = \dot{\theta}$; θ_1: the joint angle of the under-actuated link; m_1, I_{z1}, l_1: the mass, the inertia moment, and the length for the link 1 respectively; m_2, I_{z2}, l_2: the mass, the inertia moment, and the length for the link 2 respectively; r: the radius of the wheels; $2l$: the distance between two wheels; d: the distance between the manipulator and the driving center of the mobile base; m_p: the mass of the mobile platform; I_p: the inertia moment of the mobile platform; I_w: the inertia moment of each wheel; m_w: the mass of each wheel; g: gravity acceleration. The mobile manipulator is subject to the following constraint: $\dot{x}\cos\theta - \dot{y}\sin\theta = 0$. Using Lagrangian approach, we can obtain the dynamic model with $q = [\theta_l, \theta_r, \theta_1]^T$, then we could

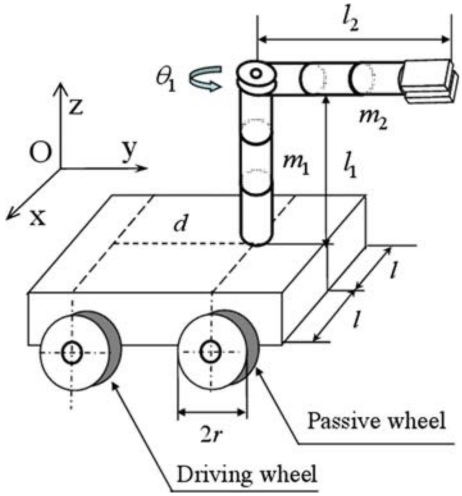


Fig. 7.1 The wheeled mobile manipulator in the simulation

obtain the dynamics as follows $M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = B\tau$. The details can be found in [31].

As discussed in Sect. 7.2, we set the fully operational configuration represented by OOO while three possible fault configurations can occur: OOF, OFO, and OFF, where O represents operational joints(or wheels) and F represents failed joints. For example, if we find that a fault occurs in τ_{θ_1}, then the fault configuration to validate the proposed methodology is the OOF configuration.

We consider a workspace with a positioning domain which ranges from $-8°$ to $12°$, with the velocities set to $1°/s$, and use 2 sectors of position in each joint, denoted as I ($-8°\sim2°$) and II ($2°\sim12°$) to map the mobile manipulator workspace. The linearization points with respect to I and II are chosen as $-3°$ and $7°$, respectively. Then, according to Sect. 7.2.3, 8 linearization points with 32 modes are found, which are shown in Table 7.2.

There exist 32 modes for the above fault-tolerant example, which means a 32×32 dimension transition rate matrix Π is needed. We may partition Π into 16 submatrices of 8×8 dimension as the following form

$$\Pi = \begin{bmatrix} \Pi_{OOO} & \Pi_{\theta_l} & \Pi_{\theta_1} & \Pi_{\theta_l,\theta_1} \\ \Pi_R & \Pi_{OFO} & \Pi_R & \Pi_{\theta_1} \\ \Pi_R & \Pi_R & \Pi_{OOF} & \Pi_{\theta_l} \\ \Pi_R & \Pi_R & \Pi_R & \Pi_{OFF} \end{bmatrix} \tag{7.22}$$

Table 7.2 The mode of operation

Mode	Joint status	Mode	Joint status	Linearization section		
				θ_r	θ_l	θ_1
1	OOO	17	OOF	I	I	I
2	OOO	18	OOF	I	II	I
3	OOO	19	OOF	I	I	II
4	OOO	20	OOF	I	II	II
5	OOO	21	OOF	II	I	I
6	OOO	22	OOF	II	II	I
7	OOO	23	OOF	II	I	II
8	OOO	24	OOF	II	II	II
9	OFO	25	OFF	I	I	I
10	OFO	26	OFF	I	II	I
11	OFO	27	OFF	I	I	II
12	OFO	28	OFF	I	II	II
13	OFO	29	OFF	II	I	I
14	OFO	30	OFF	II	II	I
15	OFO	31	OFF	II	I	II
16	OFO	32	OFF	II	II	II

where Π_{OOO}, Π_{OFO}, Π_{OOF}, and Π_{OFF} groups the relationships among the operation points in the set OOO, OFO, OOF, and OFF, respectively. Π_{θ_l} and Π_{θ_1} are related to the probability that a fault occurs in joint θ_l and θ_1, respectively, while Π_{θ_l,θ_1} represents the rate of fault occurrence in θ_l and θ_1 simultaneously. From Markov process theory, one can deduce that $\Pi_{\theta_l,\theta_1} = 0$. Π_R describes the probability that the fault in certain joint is repaired. In the mobile manipulator system, we often assume $\Pi_R = 0$, which means the defective joint cannot be repaired. From the uncertainty domain assumption (7.13), we suppose that $\Pi = \bar{\Pi} + \Delta\Pi$ and the nominal value selected heuristically as $\bar{\Pi}_{OOO}(i, i)$=-3.67, $\bar{\Pi}_{OOO}(i, j)$=0.42, $\bar{\Pi}_{OFO}(i, i)$= -2.79, $\bar{\Pi}_{OFO}(i, j)$=0.36, $\bar{\Pi}_{OOF}(i, i)$=-2.98, $\bar{\Pi}_{OOF}(i, j)$= 0.36, $\bar{\Pi}_{OFF}(i, i)$ = -1.96, $\bar{\Pi}_{OFF}(i, j)$ = 0.28, $\bar{\Pi}_{\theta_l}(i, i)$ =0.27, $\bar{\Pi}_{\theta_l}(i, j)$=0, $\bar{\Pi}_{\theta_1}(i, i)$=0.46, $\bar{\Pi}_{\theta_1}(i, j)$= 0, $\bar{\Pi}_R$=$\bar{\Pi}_{\theta_l,\theta_1}(i, j)$=$\mathbf{0}$, $\forall i, j = 1, 2, \ldots, 8,\ i \neq j$. Then we set the estimation error to 10% of the nominal values.

The system parameters can be set as $G = 0, B = I_3, I_{z1} = 1.0\,kgm^2$, $I_{z2} = 1.0\,kgm^2$, $m_1 = 1.0\,kg, m_2 = 1.1\,kg, l = 1.0\,m, l_1 = 1.0\,m, l_2 = 2.8\,m, m_p = 10.0\,kg, m_w = 2\,kg$, $I_p = 1\,Nm$, $I_w = 1.0\,kgm^2$, $r = 0.5\,m$. Then from (7.4), we get the MIMO linearized system matrices A_i, B_i, W_i, $(i = 1, 2, \ldots, 32)$ which are not listed here for economy of space. We assume the output matrix parameters are mode-independent, and set $C = [1.1I, \mathbf{0}; \mathbf{0}, \mathbf{0}]$, $D = [\mathbf{0}, \mathbf{0}; \mathbf{0}, 1.15I]$, $E = [I, \mathbf{0}]$, where $\mathbf{0}$ represents the 3×3 zero matrix. Parametric uncertainty $F(i,t),\ i = 1, 2, \ldots, 32$ is set as $F(i,t) =$ diag$[0.9\sin(it), 0.88\sin t\cos(it), 0.2\cos^2(2it), 0.3\sin(i^2 t), 0.5\cos(2i^2 t), 0.7\cos^2(it)]$ and torque disturbances $d(t)$ are introduced to verify the robustness of the controllers $d_r(t) = 0.023\sin(4t)$, $d_l(t) = 0.07\sin(3t) + 0.09\cos^2 t$ and $d_1(t) = 0.015\cos(5t)$. The disturbance is turned off after the fault introduction in corresponding joint or wheel. Other matrices E_i, L_i consisting of parametric uncertainties are set to be within an appropriate range and are not listed for details here. We choose $\varepsilon = 0.01,\ b_1 = 1.9,\ b_2 = 2.6,\ b_3 = 2.7$. The mode-independent output controller

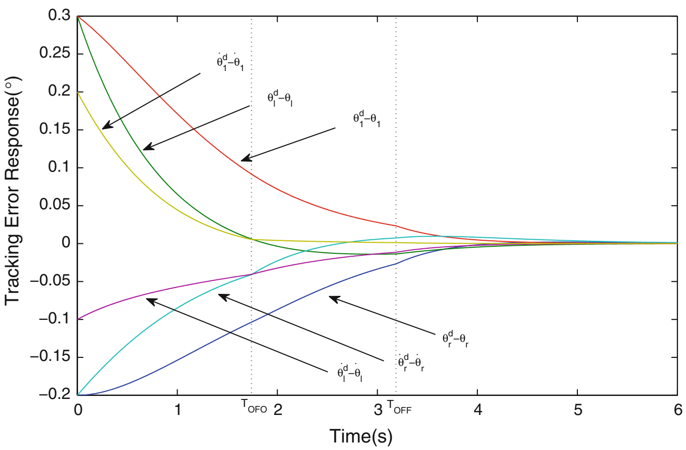


Fig. 7.2 The tracking error response

parameters in (7.14) are obtained from (7.8). We further assume that the noise attenuation level $\gamma = 1.5$ and, for simplicity and without generality, we take arbitrarily 4 modes in Table 7.2. Solving the LMIs in (7.20) while setting $\epsilon_i = 1,\ i = 1, \ldots, 4$, we obtain the mode-dependent controller gain $K_1, \ldots, K_4$. For the page limit, we omit these matrices. Figure 7.2 gives the tracking error response of $q^d - q$ and $\dot{q}^d - \dot{q}$ using the controller we get from Theorem 7.1 with the mode dependent controller gain $K_i, i = 1, \ldots, 4$ solved from LMIs (Fig. 7.3). The initial condition we used for simulation is x_0=[−0.2,0.3,0.3,−0.2,−0.1,0.2]^T, r_0=1 (Fig. 7.4).

From the simulation results, a fault first occurs in τ_l at t_{OFO}. Then another fault occurs in τ_1 at t_{OFF} so that the system mode transfers from OFO set to OFF set.

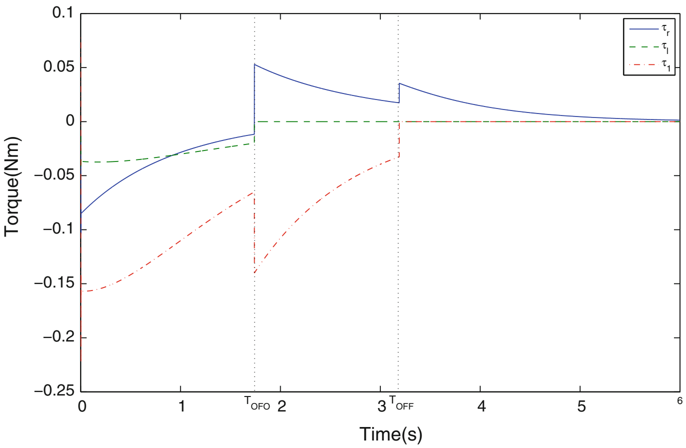


Fig. 7.3 The control signals

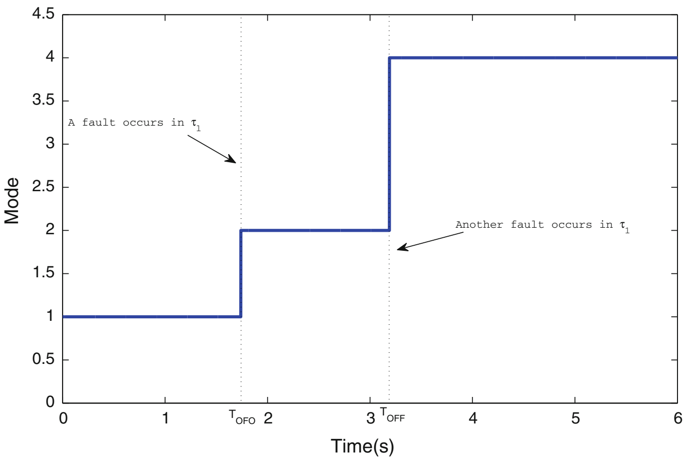


Fig. 7.4 The fault sequence

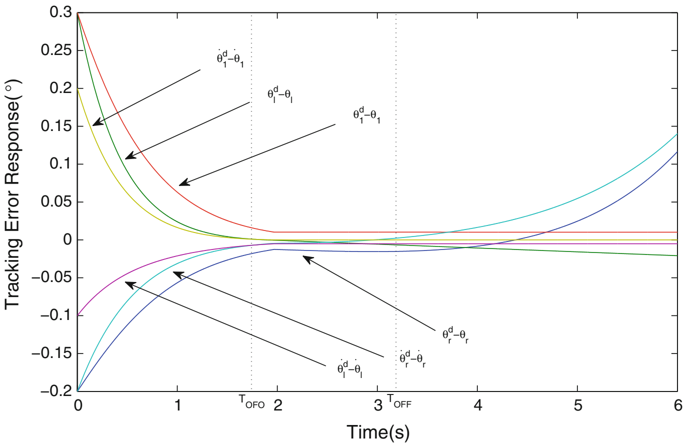


Fig. 7.5 The tracking error response with traditional controller

The tracking error decays to equilibrium point under the mode-dependent controller, which shows the fault tolerant characteristic. Meanwhile for comparison, we use a traditional output feedback controller without considering robustness and fault tolerant method as in [19]. It is obvious that the tracking performance is then unbelievable as Fig. 7.5 shows.

7.3 Optimal Control Problem of MJLS

7.3.1 An Description of Optimal Control Problem

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ be a complete probability space, in which $\{\mathscr{F}_t\}_{t\geq 0}$ is a filtration that satisfies the usual conditions. Then, consider the following MJLS:

$$\begin{aligned} \dot{x}(t) &= A(r(t))x(t) + B(r(t))u(t), \\ y(t) &= C(r(t))x(t), \\ x(t_0) &= x_0, r(t_0) = i_0, \end{aligned} \tag{7.23}$$

where $x(\cdot) \in \mathbb{R}^n$ is the continuous state, $u(\cdot) \in \mathbb{R}^m$ is the control input and $y(\cdot) \in \mathbb{R}^s$ is the output. $A(\cdot), B(\cdot), C(\cdot)$ are known system matrices with appropriate dimensions. $r(\cdot)$ is a right-continuous Markov chain taking values in a finite regime space $\mathscr{S} = \{1, 2, \ldots, N\}$.

In addition to the regime space, we also introduce an action space $\mathscr{A}$ which consists of all admissible actions. We can take any action $\alpha \in \mathscr{A}(i) \subseteq \mathscr{A}$ and apply it to the system, where $\mathscr{A}(i)$ denotes the set of actions that are available in regime

$i \in \mathscr{S}$, $\mathscr{A} = \cup_{i\in\mathscr{S}}\mathscr{A}(i)$. Here, we assume that the actions in different regime are independent. The transition rate between different regimes is determined by the action.

A stationary policy, denoted as v, can be seen as a mapping from regime space $\mathscr{S}$ to action space $\mathscr{A}$. Noting that the policy $v(i)$, $v(i) \in \mathscr{A}(i)$ determines the action in the regime i, $i \in \mathscr{S}$. Let

$$\mathscr{V} = \times_{i\in\mathscr{S}}\mathscr{A}(i) = \{(v(1), v(2), \ldots, v(N)) : v(1) \in \mathscr{A}(1), \ldots, v(N) \in \mathscr{A}(N)\}$$

be the set of all admissible policies, where "×" is called a Cartesian product, representing the direct product of sets. According to the theory in [11] and the references therein, a policy $v \in \mathscr{V}$ can be written as a vector $v : (v(1), v(2), \ldots, v(N))$. The transition rate matrix corresponding to the policy $v \in \mathscr{V}$, is denoted by $\Pi(v) = [\pi_{ij}(v(i))]_{i,j=1}^N$ where $\pi_{ij}(v(i)) \geq 0$ for $j \neq i$ and $\sum_{j\in\mathscr{S}} \pi_{ij}(v(i)) = 0$ for all $i \in \mathscr{S}$. The jumping probability between regime i and j can be described by

$$P\{r(t+\Delta) = j|r(t) = i, v(i)\} = \begin{cases} \pi_{ij}(v(i))\Delta + o(\Delta), & i \neq j \\ 1 + \pi_{ii}(v(i))\Delta + o(\Delta), & i = j \end{cases} \tag{7.24}$$

where $\Delta > 0$, $\lim\limits_{\Delta\to 0} o(\Delta)/\Delta = 0$. In this chapter, we assume that $x(t)$ and $r(t)$ can be perfectly observed at time t.

Our goal is to find a control law $(u(\cdot), v) \in \mathscr{U} \times \mathscr{V}$ such that the following cost function (or performance index) [8]

$$J(x_0, r_0, u(\cdot), v) = E\{\int_{t_0}^{\infty} [x^T(t)M(r(t))x(t) + u^T(t)N(r(t))u(t)]dt\big|x_0, r_0\}, \tag{7.25}$$

reaches its minimum, where $\mathscr{U} \times \mathscr{V}$ denotes the admissible control and policy space, $N(r(t))$ and $M(r(t))$ are positive definite and positive semi-definite matrix for any $r(t) \in \mathscr{S}$. Without loss of generality, we let $t_0 = 0$. For notational simplicity, we denote $J(u, v)$ as the above performance index, v_i the corresponding policy $v(i)$ and M_i the corresponding matrix $M(r(t) = i)$ for $i \in \mathscr{S}$.

The following definitions and assumptions are needed.

Assumption 7.3 The overall system (7.23) and (7.24) is stochastically stabilizable with admissible control and policy set $(\mathscr{U}, \mathscr{V})$. Moreover, for $C_i^T C_i = M_i$, the system (7.23) and (7.24) is stochastically observable.

Assumption 7.4 The admissible vector-valued policy space is

$$\mathscr{V} = \{v \in \mathbb{R}^N : v_{i\,\min} \leq v_i \leq v_{i\,\max},\ \ \forall i \in \mathscr{S}\}, \tag{7.26}$$

where $v_{i\,\min}$ and $v_{i\,\max}$ are given scalars. Noticing that $\mathscr{V}$ is compact and convex.

Assumption 7.5 The transition rate $\pi_{ij}(v_i)$ is continuous and smooth enough w.r.t. v_i.

Definition 7.2 Given a matrix, then, we can construct a column vector by placing the matrix's columns under each other successively. The vector is denoted as $\text{Vec}\{W_i\} \in \mathbb{C}^{mn}$. Furthermore, for all N-sequences of matrices $W = (W_1, W_2, \ldots, W_N)$ with $W_i \in \mathbb{C}^{m\times n}$, $i = 1, \ldots, N$, $\widehat{\text{Vec}}\{W\} \in \mathbb{C}^{Nmn}$ also represents a column vector by placing $\text{Vec}\{W_i\}$, $i = 1, \ldots, N$ under each other successively. Specifically, $\widehat{Vec}\{W\}$ can be written as

$$\widehat{Vec}\{W\} = \left[\text{Vec}\{W_1\}^T, \text{Vec}\{W_2\}^T, \ldots, \text{Vec}\{W_N\}^T\right]^T.$$

Remark 7.9 The transition rate defined in (7.24) indicates that the selection of the regime-dependent policy $v \in \mathscr{V}$ and the corresponding transition rate π_{ij} are not explicitly dependent on time. Take the manufacturing systems in Sect. 7.3.1 for instance, the maintenance policy or the facility layout (which is the strategy when a certain condition arises) is determined in advance and carried out at the initial time.

7.3.2 *Two-Level Regulating Method*

For the JLQ problem with the transition rate characterized by MDPs, a two-level regulating approach is employed to find a better policy. Specific steps are as follows: For the lower level, we find an optimal state feedback control law $u \in \mathscr{U}$ with a fixed transition rate control policy; For the upper level, we seek for a new transition rate control policy that has a lower cost than the present one. Iteratively, an optimal or near-optimal policy can be obtained.

7.3.2.1 The Lower Level—State-Feedback Control Law

In the lower level, the objective is to find a control law $u^* \in \mathscr{U}$ to minimize the following performance index, i.e.

$$J_{u^*}(v) \triangleq J(u^*, v) \le J(u, v), \qquad \text{for all } u \in \mathscr{U}, v \in \mathscr{V}.$$

The JLQ problem with a given policy $v \in \mathscr{V}$ is to find a state feedback control law $u = u(x, t; v)$ such that the performance index reaches its minimum. This is a reduced problem that can be solved by using the stochastic version of maximum principle(see [40]) or dynamic programming(see [5] and the references therein). Then, as in [25] or [9], the following Coupled Algebraic Riccati Equations(CARE) can be obtained.

$$A_i^T P_i + P_i A_i - P_i B_i N_i^{-1} B_i^T P_i + M_i + \sum_{i=1}^{N} \pi_{ij}(v_i) P_j = 0, \qquad \forall i \in \mathscr{S}. \quad (7.27)$$

Note here that the transition rate is policy-dependent.

Under Assumption 7.3, we obtain the following property of the solutions P_i for each $i \in \mathscr{S}$.

Lemma 7.4 *For the above reduced problem under Assumption 7.3, the set of solutions P_i of (7.27) for every $i \in \mathscr{S}$ are unique and positive definite. Furthermore, the optimal steady state control law is*

$$u^*(t) = -N_i^{-1} B_i^T P_i x(t) \tag{7.28}$$

and the overall system is stable in the mean square sense that

$$E\{x^T(t)x(t)\big|x_0, r_0\} \to 0 \quad as \;\; t \to \infty.$$

Finally, the performance index under the optimal steady state control can be calculated as follows:

$$J_{u^*}(v) = x_0^T P_{i_0} x_0. \tag{7.29}$$

Proof Following similar procedure in the proof in [25], the conclusion can be readily proved and thus the proof is omitted.

Denote the solution set by P_i. The above lemma implies that for any $i \in \mathscr{S}$, the solution P_i determines a continuous surjective mapping from the admissible policy set $\mathscr{V}$ to the set $\mathscr{P}$

$$P_i : \mathscr{V} \to \mathscr{P},$$

through the implicit constraint (7.27). Then, the performance index (7.29) is given by

$$J_{u^*}(v) = x_0^T P_{i_0}(v) x_0. \tag{7.30}$$

Notice that the above mapping corresponds to a new unique and continuous mapping from $\mathscr{V}$ to the set of the vectorized solution $\mathscr{P}_v$, i.e.

$$\mathrm{Vec}\{P_i\} : \mathscr{V} \to \mathscr{P}_v$$

7.3.2.2 The Upper Level—Near-Optimal Policy

The gradient-based method is a popular optimization algorithm with the performance gradient ∇J_{u^*} being its crux. However, the implicit relationship between v and P_i makes the traditional way of computing the gardient based on the definition $\nabla J_{u^*}(v) = \lim_{\Delta v \to 0} (J_{u^*}(v + \Delta v) - J_{u^*}(v))$ almost impossible, since the twin evil of noise and non-linearity exists [45]. The policy iterative optimization algorithm is proved to be an effective method to deal with this problem. In [45], an iterative algorithm that based on the performance gradient was proposed to find policies efficiently for the near-optimal problem with "long-run average" performance index. But for the problem with general performance index, the performance potential can not well characterize the properties of the system. In this chapter, we propose a new iterative algorithm based on the gradient projection method. We will show the effectiveness of this algorithm in dealing with the near-optimal problem with the performance index in (7.25).

Given the initial policy v_0, our goal is to reduce the value of the performance index $J_{u^*}(v)$ by tuning the policy $v \in \mathscr{V}$ in some way. In other word, we need to find a more effective policy than the initial policy v_0.

Theorem 7.3 *Under Assumption 7.3 and Assumption 7.5, the gradient of each entry of $P_i (\forall i \in \mathscr{S})$ with respect to $\mathscr{V}$ exists.*

Proof For both side of Eq. (7.27), we take the operator Vec defined in Definition 7.2, and obtain an equivalent equation

$$
\begin{aligned}
&(A_i^T \otimes I_n + I_n \otimes A_i^T)\mathrm{Vec}\{P_i\} - \frac{1}{2}(P_i B_i N_i^{-1} B_i^T \otimes I_n)\mathrm{Vec}\{P_i\} \\
&-\frac{1}{2}(I_n \otimes P_i B_i N_i^{-1} B_i^T)\mathrm{Vec}\{P_i\} + \mathrm{Vec}\{M_i\} \\
&+\sum_{i=1}^{N} \left(\pi_{ij}(v_i) \otimes I_{n^2}\right) \mathrm{Vec}\{P_j\} = 0, \qquad (7.31)
\end{aligned}
$$

where "$\otimes$" refers to the Kronecker product. An important conclusion of the operator Vec for deriving the above equation is $\mathrm{Vec}\{LKH\} = (H^T \otimes L)\mathrm{Vec}\{K\})$ with L, K, H being arbitrary real matrices.

Denote by F_i the left side of (7.31), and let $F = \left(F_1^T, F_2^T, ..., F_N^T\right)^T$. The following two statements need to be proved.

(a) F is continuously differentiable with respect to every entry of $P_i \in \mathscr{P}$ and $v_i \in \mathscr{V}, \forall i \in \mathscr{S}$.

(b) $\det(\Xi) \neq 0$, where Ξ represents the Jacobi matrix $\Xi = \partial F / \partial \left[\widehat{\mathrm{Vec}\{P\}}\right]^T$ and P represents the N-sequence matrices $(P_1, \ldots, P_N)$.

The proof of statement (a) is straightforward by using formula 7.31 and Assumption 7.5.

In order to verify the correctness of statement (b), notice that

$$\begin{aligned}\varXi &= \partial\Bigg\{\underset{i=1,2,\ldots,N}{diag}\Big[(A_i^T-\frac{1}{2}P_iB_iN_i^{-1}B_i^T)\otimes I_n+I_n\otimes(A_i^T-\frac{1}{2}P_iB_iN_i^{-1}B_i^T)\\ &\quad+\varPi(v)\otimes I_{n^2}\Big]\widehat{\mathrm{Vec}}(P)+\widehat{\mathrm{Vec}}(M)\Bigg\}/\partial\left[\widehat{\mathrm{Vec}}(P)\right]^T\\ &=\underset{i=1,2,\ldots,N}{diag}\Big[\big(A_i^T-P_iB_iN_i^{-1}B_i^T\big)\otimes I_n+I_n\otimes\big(A_i^T-P_iB_iN_i^{-1}B_i^T\big)\Big]\\ &\quad+\varPi(v)\otimes I_{n^2},\end{aligned} \tag{7.32}$$

where $M=(M_1,\ldots,M_N)$.

From the above equation, $\varXi^T$ is averaged dynamics matrix [33]. Therefore, the necessary and sufficient condition for

$$\lim_{t\to\infty}E\{x^T(t)x(t)|x_0,r_0\}=0$$

in Lemma 7.4 is that all the eigenvalues of matrix $\varXi^T$ have negative real parts, (See [32, 33] for details). Then, it gives that $\det(\varXi)\neq 0$ over the admissible policy set $\mathscr{V}$, i.e. and thus (b) is verified.

If the two statements (a) and (b) hold, then based on implicit function theorem [26], the surjective mapping $\mathrm{Vec}\{P_i\}(v)$ determined by $F=0$ is continuously differentiable on $\mathscr{V}$. That is, the existence and uniqueness of the gradient $\nabla\mathrm{Vec}\{P_i\}$ for every $v\in\mathscr{V}$ are proved.

Theorem 7.4 *Under Assumptions 7.3 and 7.5, the Hessian matrix of each entry of $P_i(\forall i\in\mathscr{S})$ with respect to $\mathscr{V}$ exists.*

Proof Theorem 7.3 implies that the surjective mapping $\mathrm{Vec}\{P_i\}(v)$ is continuously differentiable with respect to any $v_l\in\mathscr{V}, l\in\mathscr{S}$. Taking partial derivative with respect to v_l on both sides of the equation $F=0$, and then by some equivalent transformations, we obtain

$$\begin{aligned}&\Bigg\{\underset{i=1,\ldots,N}{diag}\Big[\big(A_i^T-P_iB_iN_i^{-1}B_i^T\big)\otimes I_n+I_n\otimes\big(A_i^T-P_iB_iN_i^{-1}B_i^T\big)\Big]+\varPi(v)\otimes I_{n^2}\Bigg\}\\ &\widehat{\mathrm{Vec}}\{\frac{\partial P}{\partial v_l}\}+\left(\frac{\partial\varPi(v)}{\partial v_l}\otimes I_{n^2}\right)\widehat{\mathrm{Vec}}\{P\}=0.\end{aligned} \tag{7.33}$$

where $\frac{\partial P}{\partial v_l}=(\frac{\partial P_1}{\partial v_l},\ldots,\frac{\partial P_N}{\partial v_l})$.

Denote by G_l the left side of (7.33). Then similar to (a) and (b), the correctness of the following two statements are to be verified.

(c) G_l is continuously differentiable with respect to every entry of P_i, $\partial P_i/\partial v_l$ and v_i, $\forall i\in\mathscr{S}$.

(d) $\det\left(\partial G_l/\partial\left[\widehat{\mathrm{Vec}}\{\frac{\partial P}{\partial v_l}\}\right]^T\right)\neq 0$.

Similar to Theorem 7.3, the proof for statement (c) is also straightforward.

In order to verify the correctness of statement (d), we need to deal with the following Jacobi matrix.

$$
\begin{aligned}
&\partial G_l/\partial\left[\widehat{\mathrm{Vec}}\{\frac{\partial P}{\partial v_l}\}\right]^T\\
&=\left\{\operatorname*{diag}_{i=1,\ldots,N}\left[\left(A_i^T-P_iB_iN_i^{-1}B_i^T\right)\otimes I_n+I_n\otimes\left(A_i^T-P_iB_iN_i^{-1}B_i^T\right)\right]+\Pi(v)\otimes I_{n^2}\right\}\\
&\quad\frac{\partial\widehat{\mathrm{Vec}}\{\frac{\partial P}{\partial v_l}\}}{\partial\left[\widehat{\mathrm{Vec}}\{\frac{\partial P}{\partial v_l}\}\right]^T}+0\\
&=\operatorname*{diag}_{i=1,\ldots,N}\left[\left(A_i^T-P_iB_iN_i^{-1}B_i^T\right)\otimes I_n+I_n\otimes\left(A_i^T-P_iB_iN_i^{-1}B_i^T\right)\right]+\Pi(v)\otimes I_{n^2},
\end{aligned}
$$

Let $\partial G_l/\partial\left[\widehat{\mathrm{Vec}}\{\frac{\partial P}{\partial v_l}\}\right]^T=\Xi$, then, follow similar lines as in the proof of statement (b), we can show the validity of statement (d).

Similar to Theorem 7.3, the proof is also based on implicit function theorem. Following statement (c) and (d), the implicit function theorem implies that $\mathrm{Vec}\{\partial P_i/\partial v_l\}(v)$ is continuously differentiable on $\mathscr{V}$. Then, we can show the existences and uniqueness of the Hessian $\nabla^2\mathrm{Vec}\{P_i\}$ for each $v\in\mathscr{V}$.

Both above theorems are important in the preparation of the algorithm and the corresponding convergence analysis. Next, the following projection theorem is introduced [17].

Lemma 7.5 *[17]* (The Projection Theorem for Convex Sets) *Let $x_0\in\mathbb{R}^n$ and $\Omega\subset\mathbb{R}^n$ be a nonempty closed convex set. Then $\bar{x}\in\Omega$ is the solution of the following problem*

$$\min\{||x-x_0||^2 : x\in\Omega\},$$

if and only if for any $y\in\Omega$, the inequality

$$(\bar{x}-x_0)^T(y-\bar{x})\geq 0$$

holds. Furthermore, it can be verified that the solution x always exists and is unique.

Definition 7.3 [17] Denote by $\Omega\subset\mathbb{R}^n$ a nonempty closed convex set and let $x\in\Omega$. We define a mapping $\mathrm{Proj}_\Omega:\mathbb{R}^n\to\Omega$ with

$$||\mathrm{Proj}_\Omega(x)-x||^2=\min\{||y-x||^2 : y\in\Omega\}.$$

Then, we call $\mathrm{Proj}_\Omega(x)$ is the projection of x on Ω. Lemma 7.5 implies that $\mathrm{Proj}_\Omega(x)$ is well-defined.

Denote by $\text{Proj}_{\mathscr{V}}(v)$ the projection of any policies $v \in \mathbb{R}^N$ on $\mathscr{V}$, where $\mathscr{V}$ is defined in (7.26). Noting that the policy v does not necessary belong to $\mathscr{V}$. Then, decompose the projection $\text{Proj}_{\mathscr{V}}(v)$ into each coordinate, we can obtain each components of vector $\text{Proj}_{\mathscr{V}}(v)$ by Definition 7.3.

$$[\text{Proj}_{\mathscr{V}}(v)]_i = \begin{cases} v_{i\,\min} & if \quad v_i \leq v_{i\,\min} \\ v_i & if \quad v_{i\,\min} < v_i < v_{i\,\max} \\ v_{i\,\max} & if \quad v_i \geq v_{i\,\max} \end{cases}, \tag{7.34}$$

where $[\text{Proj}_{\mathscr{V}}(v)]_i$ denotes the ith component of $\text{Proj}_{\mathscr{V}}(v)$.

In the following, we will establish our algorithm. When given an initial state x_0 and an initial regime i_0, we can calculate the gradient of the performance index (7.30) as follows.

$$\nabla J_{u^*}(v) = (I_N \otimes (x_0^T \otimes x_0^T))\left[\left(\frac{\partial \text{Vec}\{P_{i_0}\}}{\partial v_1}\right)^T, \left(\frac{\partial \text{Vec}\{P_{i_0}\}}{\partial v_2}\right)^T, ..., \left(\frac{\partial \text{Vec}\{P_{i_0}\}}{\partial v_N}\right)^T \right]^T \tag{7.35}$$

The following algorithm is proposed based on the gradient projection method [3]. In the algorithm, the superscript k represents the kth iteration.

Algorithm 7.1 *Step 1.* Set $k = 0$ and the initial stepsize $s^{(0)} > 0$. Let the small positive constant $\epsilon > 0$ denote a prescribed error margin. Suppose that the initial policy $v^{(0)} \in \mathscr{V}$ is also given.

Step 2. Given a policy $v^{(k)}$, evaluate the performance of the policy $v^{(k)}$,

- First, calculate the transition rate matrix $\Pi(v^{(k)})$ and its derivative with respect to v_l : $\frac{\partial \Pi}{\partial v_l}|_{v^{(k)}}$, $l = 1, 2, \ldots, N$;
- Then, calculate $VecP_i(v^{(k)})$, $i = 1, 2, ..., N$ by (7.27) and calculate $\frac{\partial VecP_{i_0}}{\partial v_l}|_{v^{(k)}}$, $l = 1, 2, ..., N$ by (7.33);
- Finally, substituting the result of $\frac{\partial VecP_{i_0}}{\partial v_l}|_{v^{(k)}}$, $l = 1, 2, ..., N$ in (7.35), which gives the performance gradient $\nabla J_{u^*}(v^{(k)})$;

Step 3. After evaluate the policy $v^{(k)}$, we will find a better policy $v^{(k+1)}$ according to the following equation,

$$v^{(k+1)} = \text{Proj}_{\mathscr{V}}\left[v^{(k)} - s^{(k)} \nabla J_{u^*}(v^{(k)})\right]$$

where the updating of the policy is component-wise. That is, we should update the policy for each regime based on (7.34). Another important thing is the stepsize $s^{(k)}$. Here we give the constraints the chosen stepsize should be met.

$$\lim_{k\to\infty} s^{(k)} = 0, \qquad \sum_{k=0}^{\infty} s^{(k)} = +\infty \tag{7.36}$$

Step 4. The stop condition for the iteration: If $||J_{u^*}(v^{(k+1)}) - J_{u^*}(v^{(k)})|| < \epsilon$, the algorithm stops; otherwise, set $k = k + 1$ and go to step 2.

The following theorem gives the convergence results of the proposed algorithm.

Theorem 7.5 *For the Algorithm 7.1 with Assumptions 7.3, 7.4 and 7.5, we have the convergence results,*

1. *The difference between $v^{(k+1)}$ and $v^{(k)}$ tends to 0, i.e. $\lim_{k\to\infty} ||v^{(k+1)} - v^{(k)}|| = 0$ and the limit point of $v^{(k)}$ is also a stationary point;*
2. *The infinite sequence $\{J^*(v^{(k)})\}$ decreases and converges to a finite value;*
3. *Furthermore, if $J_{u^*}(v)$ is a strongly convex function in $\mathscr{V}$, then the sequence will converge to an unique optimal policy v^*, which will minimize the performance index $J_{u^*}(v)$ in $\mathscr{V}$.*

Proof This convergence result is mainly based on Theorems 7.3 and 7.4. According to the results developed in [3, 4], the policy $v^{(k)}$, whose update is based on gradient projection algorithm with the stepsize in (7.36), converges to the stationary point v^*, and the performance index $\{J_{u^*}(v^{(k)})\}$ also converges to a finite value, if the following two conditions are met,

(i) $J_{u^*}(v)$ is continuously differentiable and bounded on $\mathscr{V}$,

(ii) the gradient $\nabla J_{u^*}(v)$ is Lipschitz continuous on any bounded subset of $\mathscr{V}$, i.e.

$$||\nabla J^*(v') - \nabla J^*(v'')|| \leq L||v' - v''|| \quad \text{for some } L > 0,\ \forall v', v'' \in \mathscr{V} \tag{7.37}$$

Because of the compactness of $\mathscr{V}$ and the conclusion that $J_{u^*}(v)$ is continuously differentiable in Theorem 7.3, we can obviously see that the condition (i) is met. For condition (ii), Theorem 7.4 indicate that the Hessian of $P_i, \forall i \in \mathscr{S}$ defined in any bounded subset of $\mathscr{V}$ is bounded in the matrix 2-norm sense. That is, $\nabla P_i(v)$ is Lipschitz continuous on any bounded subset of $\mathscr{V}$ [35]. By definition, we have the following inequality holds for any $v', v'' \in \mathscr{V}$,

$$||\nabla J^*(v') - \nabla J^*(v'')|| \leq ||x_0||^2||\nabla P_i(v') - \nabla P_i(v'')|| \leq L_0||x_0||^2||v' - v''||$$

where $L_0 > 0$ is a constant. Set $L = L_0||x_0||^2$, we have (7.37) for any bounded x_0. Therefore, Algorithm 7.1 will converge. This completes the proof.

Remark 7.10 The policy-based optimization problem is solved by Algorithm 7.1, which updates the policy recursively to obtain a better one. In Step 2, when the policy $v^{(k)}$ is given, we need to solve the CARE (7.27) to obtain $P_i(v^{(k)})$, and then, $\nabla J_{u^*}(v^k)$ can be obtained by solving the Eq. (7.33). This step seems to be the most complicated part in the algorithm. It is worth noting that once we get $P_i, i \in \mathscr{S}$, the Eq. (7.33) becomes the linear constraints with respect to $\widehat{\text{Vec}}\{\frac{\partial P}{\partial v_l}\}$, which can be solved fairly efficiently. Therefore, once we obtain the solution of the JLQ problem in the lower level, the computational cost will become relatively small.

Remark 7.11 In Algorithm 7.1, the diminishing stepsize rather than the other choice of stepsize including constant stepsize, Armijo rule, limited minimization rule, is used, mainly for the following three reasons. (1) Diminishing stepsize is an effective method when the Lipschitz constant L is unknown. We only need to verified the existences of L to guarantee the convergence. Noting that the computational cost is very high when computing L through obtaining the explicit expression of Hessian, hence the diminishing stepsize is much more efficient. (2) For some choice of stepsize, such as Armijo rule, the "stepsize judge" step is needed in each iteration, which will lead to large computational cost. This is due to the fact that each judgement is based on the variation of v, and the repetitive computation for CAREs is costly. However, this procedure is avoided for diminishing stepsize, thus the lower cost. (3) the policy space $\mathscr{V}$ is compact, which implies the boundedness of the generated policy sequence $\{v^k\}$. Then, the convergence of the algorithm with the diminishing stepsize rule is enhanced [3]).

7.3.3 Two Special Cases

In the previous subsection, we establish the general policy iterative algorithm based on the gradient projection method. However, this algorithm is no longer applicable without the initial state x_0 and initial regime r_0. In this part, we investigate two special cases, where the obtained optimal policy or near-optimal policy is more practical.

7.3.3.1 The Scalar Case

First, we consider the scalar case, where the performance index (7.30) can be rewritten as $J_{u^*}(v) = x_0^2 P_{i_0}(v)$. Suppose $\pi_{ij} > 0,\ i \neq j$, we hope that the near-optimal policy or the optimal policy obtained when $J_{u^*}(v)$ is strict convex, i.e. "strongly time consistent" [15]. That is, once a near-optimal is determined in advance, then for any $t \in [t_0, \infty)$, the policy is still near-optimal during $[t, \infty)$. That is to say, the policy is steady-state global near-optimal.

Theorem 7.6 *For the scalar system* (7.23), (7.24) *with performance index* (7.25), *the near-optimal policy has nothing to do with the initial state x_0 and the initial regime i_0.*

In the following, we give some preparations for the proof. All of the following definitions and propositions are the general theory of M-matrices [2] and gradient projection method [3].

Definition 7.4 [21] Consider a real $n \times n$ matrix $A = (a_{ij})$,

- if the off-diagonal entries are non-negative, i.e. $a_{ij} \geq 0$ for all $i \neq j$, then, we call A the Metzler matrix.

- if the off-diagonal entries are non-positive, i.e. $a_{ij} \leq 0$ for all $i \neq j$, then, we call A the Z-matrix.
- if all the eigenvalues of A are positive, then, we call A the P-matrix.
- if A is both Z-matrix and P-matrix, then, we call A the M-matrix.

Proposition 7.1 *If all the entries of M-matrix A is nonzero, then A is nonsingular and all the entries of its inverse A^{-1} are positive.*

Proposition 7.2 *A policy $v \in \mathscr{V}$ is a stationary point of $J_{u^*}(v)$ if and only if $v = \mathrm{Proj}_{\mathscr{V}}[v - t\nabla J_{u^*}(v)]$ for all $t > 0$.*

Proof of Theorem 7.6

Proof Our goal is to prove that v^* is a stationary point of $J_{1,u^*}(v) \triangleq x_1^2 P_{i_1}(v)$ in the policy space $\mathscr{V}$ for all $x_1, x_2 \in \mathbb{R}^n$ and $i_1, i_2 \in \mathscr{S}$ if and only if v^* is also a stationary point of $J_{2,u^*}(v) \triangleq x_2^2 P_{i_2}(v)$.

In light of (7.32), we can rewrite Eq. (7.33) as follows,

$$\varXi \cdot \widehat{\mathrm{Vec}}\{\frac{\partial P}{\partial v_l}\} = -\left(\frac{\partial \varPi(v)}{\partial v_l} \otimes I_{n^2}\right)\widehat{\mathrm{Vec}}\{P\}. \tag{7.38}$$

In the above equation, $\varXi$ is a Metzler matrix. We have proved in Theorem 7.3 that all the eigenvalues of matrix $\varXi$ have negative real part. Then, from Definition 7.4, $-\varXi$ is an M-matrix and nonsingular. Let

$$\varPhi = -\varXi^{-1} = [\phi_1 \ \phi_2 \ \cdots \phi_N].$$

In the following, $k_i (i \in \mathscr{S})$ denotes an N-dimensional column vector where

$$\text{the } j\text{th entry of } k_i = \begin{cases} 1 & j = i \\ 0 & otherwise \end{cases}.$$

Consider the ith entry of the vector $\widehat{\mathrm{Vec}}\{\frac{\partial P}{\partial v_l}\}$, $i \in \mathscr{S}$. For any regime $l \in \mathscr{S}$, v_l is the policy that can only apply to the corresponding regime l, then we obtain

$$\frac{\partial P_i(v)}{\partial v_l} = k_i^T \cdot \varPhi \frac{\partial \varPi(v)}{\partial v_l}\widehat{\mathrm{Vec}}\{P\} = \alpha_l k_i^T \cdot \varPhi \cdot k_l, \tag{7.39}$$

where $\alpha_l = (\frac{\partial \pi_{l1}}{\partial v_l} P_1 + \frac{\partial \pi_{l2}}{\partial v_l} P_2 + \ldots + \frac{\partial \pi_{lN}}{\partial v_l} P_N)$.

Choose arbitrary two initial regimes, i_1 and i_2. Substitute i into (7.39) with i_1 and i_2 respectively, we obtain

$$\frac{\partial P_{i_1}(v)}{\partial v_l}\frac{\partial P_{i_2}(v)}{\partial v_l} = \frac{\partial P_{i_1}(v)}{\partial v_l}\left[\frac{\partial P_{i_2}(v)}{\partial v_l}\right]^T = \alpha_l^2 k_{i_1}^T \varPhi k_l \cdot k_l^T \varPhi^T k_{i_2} = \alpha_l^2 k_{i_1}^T [\phi_l \phi_l^T] k_{i_2} \tag{7.40}$$

Proposition 7.1 implies that for every $l \in \mathscr{S}$, each entry of matrix ϕ_l is positive. Together with (7.39) and (7.40), the following two statements hold.

(e) $\frac{\partial P_{i_1}(v)}{\partial v_l}\frac{\partial P_{i_2}(v)}{\partial v_l} \geq 0$,

(f) $\frac{\partial P_{i_1}(v)}{\partial v_l} = 0$ if and only if $\frac{\partial P_{i_2}(v)}{\partial v_l} = 0$.

Suppose that v^* is a stationary point of $J_{1,u^*}(v) = x_1^2 P_{i_1}(v)$. Then, we derive the gradient of J_{1,u^*} and J_{2,u^*} at v^*,

$$\nabla J_{1,u^*}(v^*) = x_1^2 \nabla P_{i_1}(v^*), \quad \nabla J_{2,u^*}(v^*) = x_2^2 \nabla P_{i_2}(v^*), \tag{7.41}$$

where $x_1, x_2 \in \mathbb{R}^n$ are the initial states. From the system dynamics (7.23) and the control law (7.28), we can conclude that x_1 and x_2 are nonzero.

According to statements (e) and (f), we can verify that there exists a positive constant d_l such that the following equality holds.

$$\left.\frac{\partial P_{i_2}(v)}{\partial v_l}\right|_{v^*} = d_l \left.\frac{\partial P_{i_1}(v)}{\partial v_l}\right|_{v^*}.$$

By the above equality and formula (7.41), we obtain

$$\nabla J_{2,u^*}(v^*) = \theta x_1^2 D \nabla P_{i_1}(v^*) = \theta D \nabla J_{1,u^*}(v^*), \tag{7.42}$$

where $\theta = (x_2/x_1)^2 > 0,\ D = \underset{l=1,2,\ldots,N}{diag}\{d_l\},\ d_l > 0$.

From Proposition 7.2, we see that for all $t > 0$, each component of v^* satisfies the following equation,

$$v_i^* = \text{Proj}_{\mathscr{V}}[v_i^* - t \left.\frac{\partial J_{1,u^*}(v)}{\partial v_i}\right|_{v^*}],$$

Consider the optimality condition [3], if v^* is a stationary point of $J_{1,u^*}(v)$ in $\mathscr{V}$, then

$$\left.\frac{\partial J_{1,u^*}(v)}{\partial v_i}\right|_{v^*} = 0, \quad \text{if } v_{i\,\min} < v_i^* < v_{i\,\max}, \tag{7.43}$$

$$\left.\frac{\partial J_{1,u^*}(v)}{\partial v_i}\right|_{v^*} \geq 0, \quad \text{if } v_i^* = v_{i\,\min}, \tag{7.44}$$

$$\left.\frac{\partial J_{1,u^*}(v)}{\partial v_i}\right|_{v^*} \leq 0, \quad \text{if } v_i^* = v_{i\,\max}. \tag{7.45}$$

For the projection of $\left[v_i^* - t \left.\frac{\partial J_{2,u^*}(v)}{\partial v_i}\right|_{v^*}\right]$ when $t > 0$, three cases are considered.

1. $v_{i\,\min} < v_i^* < v_{i\,\max}$.

The optimality condition (7.43) indicates that $\frac{\partial J_{1,u^*}(v)}{\partial v_i}\Big|_{v^*} = 0$, and the statement (e) indicates that $\frac{\partial J_{1,u^*}(v)}{\partial v_i}\Big|_{v^*} = 0$ if and only if $\frac{\partial J_{2,u^*}(v)}{\partial v_i}\Big|_{v^*} = 0$. Then, we can conclude that

$$\mathrm{Proj}_{\mathscr{V}}[v_i^* - t\frac{\partial J_{2,u^*}(v)}{\partial v_i}\Big|_{v^*}] = v_i^*$$

2. $v_i^* = v_{i\,\min}$.

The optimality condition (7.44) indicates that $\frac{\partial J_{1,u^*}(v)}{\partial v_i}\Big|_{v^*} \geq 0$. Considering (7.42) and notice $t > 0, \theta > 0, d_i > 0$, we obtain

$$v_i^* - t\frac{\partial J_{2,u^*}(v)}{\partial v_i}\Big|_{v^*} = v_i^* - t\theta d_i\frac{\partial J_{1,u^*}(v)}{\partial v_i}\Big|_{v^*} \leq v_i^* = v_{i\,\min}$$

From the definition of the projection on $\mathscr{V}$ in (7.34), we can conclude that

$$\mathrm{Proj}_{\mathscr{V}}[v_i^* - t\frac{\partial J_{2,u^*}(v)}{\partial v_i}\Big|_{v^*}] = v_{i\,\min} = v_i^*.$$

3. $v_i^* = v_{i\,\max}$.

Take a similar line as the above case, since $\frac{\partial J_{1,u^*}(v)}{\partial v_i}\Big|_{v^*} \leq 0, t > 0, \theta > 0, d_i > 0$. we have

$$v_i^* - t\frac{\partial J_{2,u^*}(v)}{\partial v_i}\Big|_{v^*} = v_i^* - t\theta d_i\frac{\partial J_{1,u^*}(v)}{\partial v_i}\Big|_{v^*} \geq v_i^* = v_{i\,\max},$$

Then, by (7.34), we also can conclude that

$$\mathrm{Proj}_{\mathscr{V}}[v_i^* - t\frac{\partial J_{2,u^*}(v)}{\partial v_i}\Big|_{v^*}] = v_{i\,\max} = v_i^*.$$

The above conclusions indicate that for any $t > 0$, we have

$$v^* = \mathrm{Proj}_{\mathscr{V}}[v^* - t\nabla J_{2,u^*}(v^*)] \tag{7.46}$$

In light of Proposition 7.2, (7.46) is equivalent to the condition that $v^* \in \mathscr{V}$ is a stationary point of $J_{2,u^*}(v)$. Notice that x_1, x_2, i_1, i_2 are all chosen arbitrarily, we then complete the proof of Theorem 7.6.

By the conclusion in Theorem 7.6, Algorithm 7.1 can be simplified by setting the initial state and initial regime as $x_0 = 1$ and $i_0 = 1$ to obtain a near-optimal policy,

Algorithm 7.2 *Step 1.* Set $k = 0$ and the initial stepsize $s^{(0)} > 0$. Let $\epsilon > 0$ be the prescribed error margin. Supposed that the initial policy $v^{(0)} \in \mathscr{V}$ is also given.
Step 2. Given a policy $v^{(k)}$, evaluate the performance of this policy,

- First, calculate the transition rate matrix $\Pi(v^{(k)})$ and its derivative with respect to $v_l : \frac{\partial \Pi}{\partial v_l}|_{v^{(k)}},\ l = 1, 2, \ldots, N$;
- Then, calculate $P_i(v^{(k)}),\ i = 1, 2, ..., N$ by (7.27);
- Finally, calculate $\frac{\partial P_1}{\partial v_l}|_{v^{(k)}}$, $l = 1, 2, ..., N$ by (7.33)

Step 3. After evaluating the policy $v^{(k)}$, we will find a better policy $v^{(k+1)}$ according to the following equation,

$$v^{(k+1)} = \text{Proj}_{\mathscr{V}} \left[v^{(k)} - s^{(k)} \nabla P_1(v^{(k)}) \right].$$

where the updating of the policy is component-wise. That is, we should update the policy for each regime based on (7.34). Besides, we choose the stepsize $s^{(k)} > 0$ which meet the following constraints

$$\lim_{k \to \infty} s^{(k)} = 0, \qquad \sum_{k=0}^{\infty} s^{(k)} = +\infty. \tag{7.47}$$

Step 4. The stop condition for the iteration:
If $||P_1(v^{(k+1)}) - P_1(v^{(k)})|| < \epsilon$, the algorithm stops; otherwise, set $k = k + 1$ and go to step 2.

Remark 7.12 Notice that we need the assumption $\pi_{ij} > 0,\ i \neq j$ such that Proposition 7.1 can be used in the proof of Theorem 7.6. Here, It is worth pointing out that this constraint can be released. For instance, from the M-matrix theory [2], if an M-matrix with some zero entries is irreducible, then all the entries of its inverse are positive.

Remark 7.13 In the scalar case, a near-optimal policy without knowing the initial state and regime can be found if the system parameters remain unchanged. The advantage mainly lie in the fact that even the system is in operation, one can still seek for a near-optimal policy by Algorithm 7.1, and then apply the policy to the system at any time. This can be seen as an "on-line decision" rather than the "off-line decision" as interpreted in Remark 7.9. However, for the non-scalar case, this property may not hold any more.

7.3.3.2 The Case with Unknown Initial State

The conclusion in Theorem 7.6 is not applicable to multiple dimensional systems. In this part we consider the case where the initial state x_0 is unknown. By assuming that x_0 is a random variable with known statistical properties, we can decouple x_0 from the control law. This method has been used to deal with other control problem, see, [24, 28, 36].

Assumption 7.6 The initial state x_0 is a zero-mean random variable with covariance $E\{x_0 x_0^T\} = \sigma^2 I_n$.

Under the above assumption, $J_{u^*}(v) = x_0^T P_{i_0} x_0$ becomes a random variable. So we can use the expectation $\hat{J}_{u^*}(v) = E\{x_0^T P_{i_0} x_0\}$ to characterize the performance. Then, we have

$$\hat{J}_{u^*}(v) = E\{x_0^T P_{i_0} x_0\} = tr[P_{i_0} E\{x_0 x_0^T\}] = tr[P_{i_0} \sigma^2 I_n] = \sigma^2 tr[P_{i_0}] \tag{7.48}$$

and the corresponding gradient is

$$\nabla \hat{J}_{u^*}(v) = \sigma^2 [tr(\frac{\partial P_{i_0}}{\partial v_1}), tr(\frac{\partial P_{i_0}}{\partial v_2}), \ldots, tr(\frac{\partial P_{i_0}}{\partial v_N})]^T. \tag{7.49}$$

In the following, we establish an algorithm to seek for a near-optimal policy for this case.

Algorithm 7.3 *Step 1*. Set $k = 0$ and the initial stepsize $s^{(0)} > 0$. Let the small positive constant $\epsilon > 0$ denote the prescribed error margin. Suppose that the initial policy $v^{(0)} \in \mathscr{V}$ is also given.
Step 2. Given a policy $v^{(k)}$, evaluate the performance of this policy,

- First, calculate the transition rate matrix $\Pi(v^{(k)})$ and its derivative with respect to $v_l : \frac{\partial \Pi}{\partial v_l}|_{v^{(k)}},\ l = 1, 2, \ldots, N$
- Then, calculate $VecP_i(v^{(k)}), i = 1, 2, ..., N$ by (7.27) and calculate $\frac{\partial VecP_{i_0}}{\partial v_l}|_{v^{(k)}}$, $l = 1, 2, ..., N$ by (7.33);
- Filially, calculate the performance gradient $\nabla \hat{J}_{u^*}(v^{(k)})$ by (7.49);

Step 3. After evaluate the policy $v^{(k)}$, we will find a better policy $v^{(k+1)}$ according to the following equation,

$$v^{(k+1)} = \text{Proj}_{\mathscr{V}} \left[v^{(k)} - s^{(k)} \nabla \hat{J}_{u^*}(v^{(k)}) \right],$$

where the updating of the policy is component-wise. That is, we should update the policy for each regime based on (7.34). Besides, we choose the stepsize $s^{(k)}$ which meet the following constraints:

$$\lim_{k \to \infty} s^{(k)} = 0, \qquad \sum_{k=0}^{\infty} s^{(k)} = +\infty. \tag{7.50}$$

Step 4. The stop condition for thr iteration:

If $||\hat{J}_{u^*}(v^{(k+1)}) - \hat{J}_{u^*}(v^{(k)})|| < \epsilon$, the algorithm stops; otherwise, set $k = k + 1$ and go to step 2.

The convergence analysis is similar as Theorem 7.5, thus is omitted here.

Remark 7.14 With Assumption 7.6, the modified performance index is a scalar rather than the random variable, which is desirable. Noting that this assumption is not general, so examining its rationality may be needed for any practical problem.

7.3.4 Numerical Simulation

In this section, we demonstrate the developed results by two examples. In the first example, a near-optimal policy found by Algorithm 7.1 can effectively improve the performance. The second example considers the scalar case where the near-optimal policy is independent on the initial state and the initial regime.

Example 7.1 Consider a two-dimension MJLS with two regimes. Let

$$A_1 = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 0.6 \\ 0.2 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 1 \\ -0.1 \end{bmatrix}$$
$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$N_1 = 1.4 \qquad N_2 = 2.1$$
$$\Pi(v) = \begin{bmatrix} -2e^{-v_1} & 2e^{-v_1} \\ e^{-v_2} & -e^{-v_2} \end{bmatrix}$$

In this example, some prescribed parameters are given as $\epsilon = 1 \times 10^{-5}$, $v^{(0)} = [0.3\ 0]^T$, $s^{(k)} = 0.05 \times k^{-1/3}$ for all k. Denote the admissible policy space by

$$\mathscr{V} = \{v \in \mathbb{R}^N : -2 \le v_i \le 2, \quad i = 1, 2\}. \tag{7.51}$$

Assume the initial state and initial regime to be $x_0 = [-1, 0.5]^T$, $r_0 = 1$. Then, applying Algorithm 7.1, a near-optimal policy $v^* = [-0.4917, 2]$ can be obtained within 50 iterations. The performance curve during the iteration procedure is illustrated in Fig. 7.6. Finially, the performance is $J^* = 77.5941$ which is about 26% better than the original one.

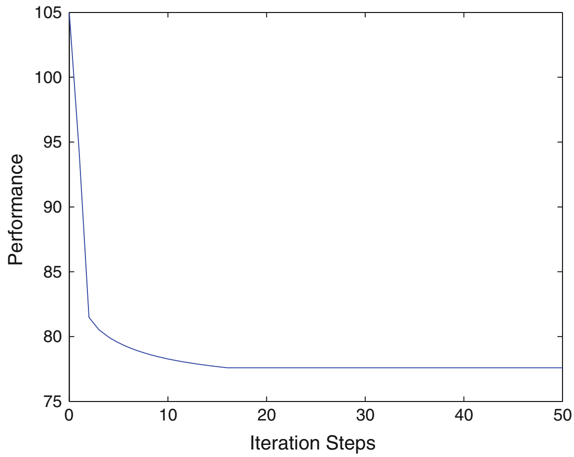


Fig. 7.6 Performance improvement (Non-Scalar Case)

Example 7.2 Consider a two-regime scalar MJLS where

$$A_1 = 1.3, A_2 = 1, B_1 = 1, B_2 = 1.2,$$
$$M_1 = 1.1, M_2 = 2, N_1 = 1.2, N_2 = 0.8.$$

and

$$\Pi(v) = \begin{bmatrix} -0.4v_1^2 e^{v_1} & 0.4v_1^2 e^{v_1} \\ 3e^{v_2^2} & -3e^{v_2^2} \end{bmatrix}$$

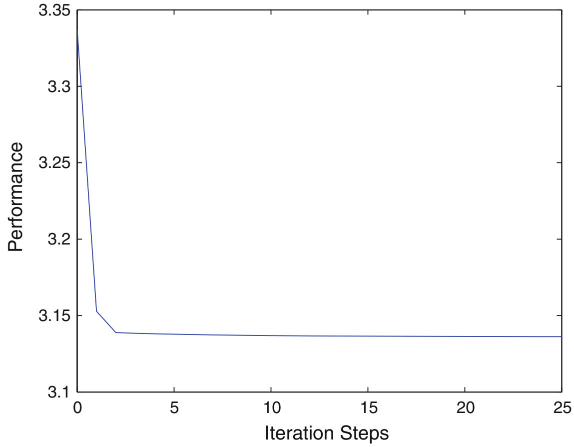
In this example, parameters are given as $\epsilon = 1 \times 10^{-5}$, $v^{(0)} = [0.7\ 0.2]^T$, $s^{(k)} = 0.6 \times k^{-1/4}$ for all k. Denote the admissible policy space by

$$\mathscr{V} = \{v \in \mathbb{R}^N : -1 \leq v_i \leq 1, \quad i = 1, 2\}. \tag{7.52}$$

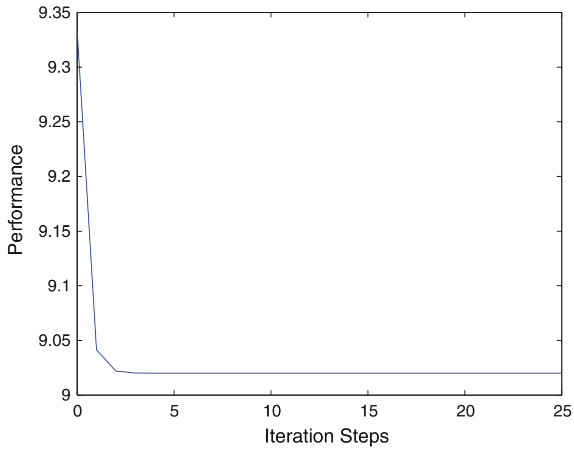
Assume the initial state and initial regime to be $x_0 = 1, r_0 = 1$, then a near-optimal policy $v^* = [1, 0]$ is obtained. The performance converges to $J^* = 3.1362$. Figure 7.7a shows the performance curve during the iteration procedure. Next, the initial state and initial regime are changed to $x_0 = 2, r_0 = 2$, then the near-optimal policy remains $v^* = [1, 0]$ and the corresponding performance converges to $J^* = 9.0201$. This procedure is shown in Fig. 7.7b.

Due to the simplicity of the example we can describe the performance index on the policy space completely. The two cases are shown in Figs. 7.8a, b, respectively. Although the performance is changed with the initial state and initial regime, the near-optimal policy remains unchanged.

Fig. 7.7 Performance improvement for Scalar Case



(a) Initial condition:$x_0 = 1, r_0 = 1$



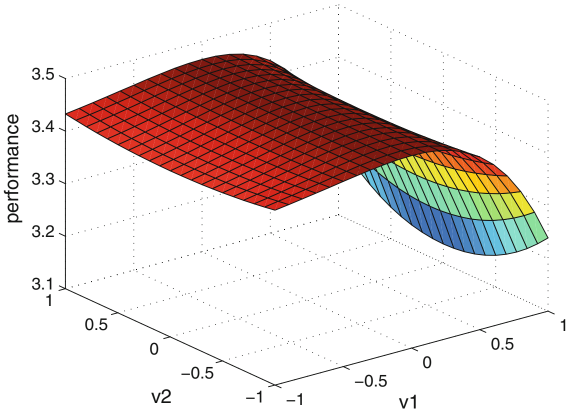
(b) Initial condition:$x_0 = 2, r_0 = 2$

7.4 Summary

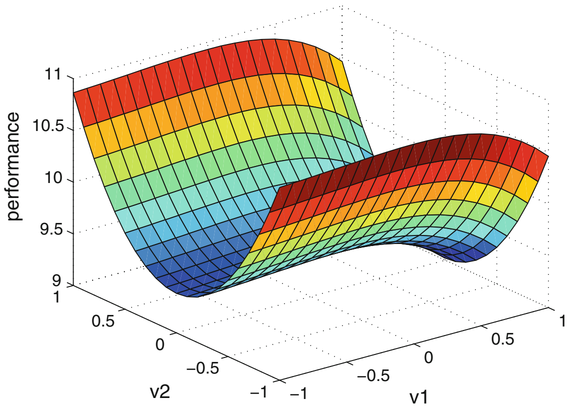
In the first part of this chapter, a Markovian fault-tolerant model is developed for a mobile manipulator with two independent wheels and multiple joints. The uncertainty of the transition rate matrix is considered in an element-wise way. We have presented sufficient conditions on the existence of mode-dependent dynamic output feedback control based on a high-gain observer.

The second part deals with the JLQ problem of a class of MJLS whose regime transition rates are determined by the initial policy. Based on the two-level regulating method and the gradient projection method, an algorithm is proposed to seek a near-optimal policy, and the convergence result of the algorithm is also developed. If

Fig. 7.8 Performance over the policy space for Scalar Case



(a) Initial condition:$x_0 = 1, r_0 = 1$



(b) Initial condition:$x_0 = 2, r_0 = 2$

the property of concavity is unknown, some special method, including simulated annealing [42], quantum annealing, Tabu search, can be employed to find a global optimum policy. These will be our future works.

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