Leader-Following Cluster Consensus of Multiagent Systems With Measurement Noise and Weighted Cooperative–Competitive Networks

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***Abstract*—Leader-following cluster consensus is investigated for multiagent systems with weighted cooperative–competitive networks and measurement noise. A stochastic approximation protocol is proposed for interactively balanced and sub-balanced networks, and pinning control is introduced to deal with the divergence phenomenon in interactively unbalanced networks. With these protocols, sufficient conditions for reaching a strong mean-square leader-following cluster consensus are established for all the three types of networks, which are also extended to the cases without measurement noise. Numerical examples illustrate the effectiveness of the proposed protocols and theoretical analysis.**

***Index Terms*—Cluster consensus, leader-following, measurement noise, multiagent systems (MASs), weighted cooperative–competitive networks.**

I. Introduction

CONSENSUS has long been a central topic to multiagent systems (MASs), receiving much attention from the research community [1]–[11]. In [1], the notion of consensus is first proposed for single-integrator agents under a directed network. A type of linear Laplacian feedback consensus protocol is designed for agents under fixed and switching topologies. It is shown that the directed network being strongly connected is sufficient for ensuring consensus. The network condition is further relaxed in [2] to the case that the directed network having a spanning tree. Then, the existence of consensus protocols, called consensusability, is studied in [3]. It is pointed out that consensusability of an MAS is closely related to three key factors, i.e., the dynamics of each agent, communication interactions among agents, and the admissible control set. Necessary and sufficient consensusable conditions on the above three factors are presented for the general linear MASs. Further, the consensusability problem in [3] is extended in [5] to the discrete-time model over analog fading networks. And then, optimal consensus for uncertain nonlinear MASs is addressed in [10], where the proposed two fully distributed adaptive protocols with disturbance rejection can ensure consensus with minimized local cost functions. It is noted that consensus is conventionally interpreted as the scenario where all agents reach one common value. However, in recent years, it is revealed that at least for certain MASs, the agents may not necessarily converge uniformly to one common value, but reach consensus in a "clustered" fashion, i.e., there may exist more than one cluster of agents where the agents in each cluster can reach consensus in the conventional sense, as seen in foraging activities with mixed specifies [12], social networks [13], etc. This type of consensus is often referred to as "cluster consensus," which can be in the form of either leaderless [14]–[17] or leader-following [18]–[23]. In this work, we are particularly interested in the latter form, one typical example of which is the honeybee swarms. It is reported that approximately 5% of the honeybees, who are regarded as the "leaders," can guide all the bees, which include all the other honeybees regarded as the "followers," to a new nest site [24].

Leader-following cluster consensus has received increased interest in recent years. To name a few, in [18], such consensus is considered for second-order nonlinear MASs, with criteria being given to ensure asymptotical cluster consensus. The continuous second-order dynamics in [18] is then extended to the discrete-time version in [19]. In [20], leader-following cluster consensus for double-integrator MASs is analyzed under two different topology frameworks. It is proven that for both frameworks the cluster consensus can be guaranteed under conditions on coupling strengths of the considered topology. In [21], the leader-following practical cluster consensus for generic linear MASs is studied, where an event-triggered protocol is designed for each follower, and it is shown that cluster consensus can be ensured by selecting proper parameters in protocols regardless of the state estimation. Furthermore, a frequency domain method is employed in [22] to investigate the leader-following cluster consensus for heterogeneous MASs.

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However, existing works on leader-following cluster consensus need further developments in the following two aspects.

First, all the above-mentioned works are based on cooperative networks or binary cooperative–competitive networks, where all the agents are either cooperative for the former or both cooperative and competitive for the latter, represented by a signed digraph using non-negative and negative edge weights for cooperative and competitive interactions, respectively. Generally, the relationship among agents can be far more complex than the context of either cooperation or competition. For instance, in the process of opinion formation, one person may put different weights of approval or disapproval on others' opinions. This inspired the weighted cooperative–competitive networks as introduced in [25], where a weight matrix is employed to denote the extent of cooperation/competition among agents and the network is divided into three types.

Second, existing studies usually assume the accurate information acquisition for all the agents, which can be practically impossible in many cases due to the existence of measurement noise. Despite the works on measurement noise for consensus [26]–[31] and bipartite consensus [32]–[34], only a few results on cluster consensus [35]–[36] still assume binary cooperative–competitive networks, while not surprisingly weighted cooperative–competitive networks can be more challenging.

To meet the above challenges, this work considers leader-following cluster consensus of MASs perturbed by measurement noise under weighted cooperative–competitive networks, where the network is divided into three types: 1) interactively balanced network; 2) sub-balanced network; and 3) unbalanced network.

1) Knowing that the noise interference and the coupling states of agents are the main difficulties for interactively balanced and sub-balanced networks, we propose a distributed protocol with a time-varying function gain. The time-varying gain then yields the closed-loop system in the form of a challenging time-varying stochastic differential equation. We prove the state convergence of the followers to the leaders under the proposed protocol, with a constant proportion in the strong mean-square sense, thus the leader-following cluster consensus.
2) We realize the fact that there exists at least one directed circle with the weight product being not 1 for interactively unbalanced networks, which then readily leads to divergence. To deal with this phenomenon, we design a pinning controller for certain followers to ensure leader-following cluster consensus in the strong mean-square sense. We also extend the results with measurement noise to the case without it. Sufficient conditions are provided with regard to the time-varying function gain and pinning control to design leader-following cluster protocols.

The main contributions of this work are threefold. First, both measurement noise and weighted cooperative–competitive network are considered, which reflect a more practical model in many cases. Second, pinning control is introduced to address the strong mean-square leader-following cluster consensus problem for interactively unbalanced networks, which is technically novel. Finally, with the help of stochastic analysis and graph theory, the proposed stochastic approximation protocols successfully ensure leader-following cluster consensus, regardless of measurement noise.

The remainder of this work is organized as follows. Section II reviews the results of algebraic graph theory and then introduces the considered problem. The leader-following cluster consensus for the considered system is discussed under three kinds of weighted cooperative–competitive networks in Section III, respectively. Section IV validates the theoretical results using numerical simulations, and Section V concludes this article.

Notations: $\mathbf{1}_n = (1, \ldots, 1)^T$ is a column vector with n-dimension. $\mathbf{0}$ is a matrix (vector) with an appropriate dimension. $\text{diag}(a_1, \ldots, a_n)$ is a diagonal matrix with diagonal elements $a_i, i = 1, \ldots, n$. $\lambda_{\max}(B)$ and $\lambda_{\min}(B)$ denote the maximum and the minimum eigenvalues of a symmetric matrix B, respectively. $a \wedge b$ denotes the smaller one between a and b.

II. Preliminaries

A. Algebraic Graph Theory

Communication interactions among agents are described by a weighted cooperative–competitive network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$, where $\mathcal{V}$ is the node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{n\times n}$ is the adjacency matrix, $a_{ij} \geq 0$ and $a_{ij} > 0$ iff $(j, i) \in \mathcal{E}$. $\mathcal{D} = (d_{ij}) \in \mathbb{R}^{n\times n}$ is a weight matrix, d_{ij} represents the weight extent among agents and $d_{ij} \neq 0$ iff $(j, i) \in \mathcal{E}$. Furthermore, $d_{ij} > 0$ represents the cooperative extent of i and j; $d_{ij} < 0$ represents the competitive extent of i and j. We assume there exists no self-loop. Agent i's neighbor set is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$. $L = (l_{ij})$ is the Laplacian matrix of $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$, where $l_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$, $l_{ij} = -a_{ij}d_{ij}, i \neq j$. Let $L_s = (l_{s,ij})_{n\times n}$ be the standard Laplacian matrix with $l_{s,ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$, $l_{s,ij} = -a_{ij}$.

A directed path in $\mathcal{G}$ is a sequence of edges $(i, \mathcal{V}_1)(\mathcal{V}_1, \mathcal{V}_2) \cdots (\mathcal{V}_k, j)$, expressed as $\mathcal{P}_{ij}$, where i, $\mathcal{V}_1, \ldots, \mathcal{V}_k, j$ are distinct nodes. The interaction weight product of this path is $R_{ij} = d_{\mathcal{V}_1 i} d_{\mathcal{V}_2 \mathcal{V}_1} \cdots d_{j \mathcal{V}_k}$. A directed path $\mathcal{P}_{ij}$ becomes a directed cycle $\mathcal{C}_i$ if $i = j$, and hence $R_{ii} \triangleq C_i = d_{\mathcal{V}_1 i} d_{\mathcal{V}_2 \mathcal{V}_1} \cdots d_{i \mathcal{V}_k}$. The bidirectional graph of $\mathcal{G}$ is denoted by $\hat{\mathcal{G}} = (\mathcal{V}, \hat{\mathcal{E}}, \hat{\mathcal{A}}, \hat{\mathcal{D}})$, where $\hat{\mathcal{A}} = (\hat{a}_{ij})_{n\times n}$ and $\hat{\mathcal{D}} = (\hat{d}_{ij})_{n\times n}$ such that

$$\hat{a}_{ij} = \begin{cases} a_{ji}, & (j, i) \notin \mathcal{E}, (i, j) \in \mathcal{E} \\ a_{ij}, & \text{else} \end{cases}$$

$$\hat{d}_{ij} = \begin{cases} d_{ji}^{-1}, & (j, i) \notin \mathcal{E}, (i, j) \in \mathcal{E} \\ d_{ij}, & \text{else.} \end{cases}$$

$\mathcal{G}$ is interactively balanced if all the weight products of directed circles in $\hat{\mathcal{G}}$ are equal to 1. $\mathcal{G}$ is interactively sub-balanced if $\mathcal{G}$ does not have directed cycles and $\exists C_i \neq 1$ in $\hat{\mathcal{G}}$. $\mathcal{G}$ is interactively unbalanced if $\mathcal{G}$ has directed cycles and $\exists C_i \neq 1$ in $\hat{\mathcal{G}}$.

Some useful lemmas are first provided, as follows.

Lemma 1 [25]: Given $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$ with a spanning tree. $\mathcal{G}$ is interactively balanced if there exists an invertible matrix $R = \mathrm{diag}(R_{11}, \ldots, R_{1n})$ satisfying $R^{-1}LR = L_s$.

Lemma 2 [2]: Let L_s be a standard Laplacian matrix in cooperative network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. L_s has a single eigenvalue equal to zero and all other eigenvalues have positive real parts iff $\mathcal{G}$ contains a spanning tree.

B. Problem Formulation

Consider an MAS of one leader and N followers, where the leader is labeled by 0 and followers are labeled by $1, \ldots, N$, respectively. The state of the ith follower evolves according to the following system:

$$\dot{x}_i(t) = u_i(t) \tag{1}$$

where x_i, $u_i \in \mathbb{R}$ are the state and control input. The leader's state is not affected by the followers and hence is assumed to be a constant, denoted by x_0.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$ be the communication network of one leader and N followers. Denote the communication network of N followers as $\mathcal{G}_f = (\mathcal{V}_f, \mathcal{E}_f, \mathcal{A}_f, \mathcal{D}_f)$. Then

$$\begin{aligned} \mathcal{V}_f &= \{1, \ldots, N\}, \\ \mathcal{V} &= \{0\} \bigcup \mathcal{V}_f, \\ \mathcal{A} &= \begin{pmatrix} 0 & 0 \\ G_s \cdot \mathbf{1_N} & \mathcal{A}_f \end{pmatrix}, \\ \mathcal{D} &= \begin{pmatrix} 0 & 0 \\ \mathcal{D}_0 \cdot \mathbf{1_N} & \mathcal{D}_f \end{pmatrix} \end{aligned}$$

where $G_s = \mathrm{diag}(g_1, \ldots, g_N)$, $g_i > 0$ iff $0 \in \mathcal{N}_i$, $i = 1, \ldots, N$, and $\mathcal{D}_0 = \mathrm{diag}(d_{10}, \ldots, d_{N0})$. The Laplacian of $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$ is

$$L = \begin{pmatrix} 0 & 0 \\ -G_s\mathcal{D}_0 \cdot \mathbf{1_N} & L_{\mathcal{G}_f} + G_s \end{pmatrix} \tag{2}$$

where $L_{\mathcal{G}_f}$ is the Laplacian of $\mathcal{G}_f$.

Due to the measurement noise, the ith follower receives information from its neighbors in the following form:

$$x_{ji}(t) = x_j(t) + \beta_{ji}(t), \quad j \in \mathcal{N}_i, \quad i = 1, \ldots, N$$

in which $x_{ji}(t)$ is the measurement of the jth agent's state $x_j(t)$ by the ith agent, and $\{\beta_{ji}(t), j \in \mathcal{N}_i, i = 1, \ldots, N\}$ are independent standard white noise.

The primary objective of this work is to introduce a distributed protocol for each follower subject to measurement noise such that the followers achieve cluster consensus to the leader.

The following cluster consensus protocol is used:

$$\begin{aligned} u_i(t) = f(t) \sum_{j \in \mathcal{N}_i} \Big[a_{ij}\big(d_{ij}x_{ji}(t) - x_i(t)\big) \\ + g_i(d_{i0}x_{0i}(t) - x_i(t))\Big], i = 1, \ldots, N \end{aligned} \tag{3}$$

where $f(t) > 0$ is a piecewise continuous function.

Clearly, protocol (3) is based on the states of agent i and its neighbors and thus is distributed.

Remark 1:

1) Different from leader-following cluster consensus in [18]–[22], the effect of measurement noise is considered in protocol (3). To deal with the measurement noise, a time-varying function gain $f(t)$ is employed here.
2) If $d_{ij} \equiv 1 (j \in \mathcal{N}_i)$, then the protocol in (3) is reduced to the protocol in [28]; if $d_{ij} \in \{\pm 1\} (j \in \mathcal{N}_i)$, then the protocol in (3) is reduced to the protocol in [34].

Let $X(t) = (x_1(t), \ldots, x_N(t))^T$. Applying the protocol in (3) to the system in (1), one obtains

$$\begin{aligned} dX(t) = -f(t)\Big(L_{\mathcal{G}_f} + G_s\Big)X(t)dt + f(t)G_s\mathcal{D}_0 \cdot \mathbf{1}_N x_0 dt \\ + f(t)QdW^*(t) \end{aligned} \tag{4}$$

where $Q = (H, G_s\mathcal{D}_0)$ is an $N \times (N^2 + N)$-dimensional matrix with $H = \mathrm{diag}(h_1^T, \ldots, h_N^T) \in \mathbb{R}^{N \times N^2}$ and $h_i^T = (a_{i1}d_{i1}, \ldots, a_{iN}d_{iN})$. $W^*(t) = (w_{11}(t), w_{21}(t), \ldots, w_{N1}(t), \ldots, w_{1N}(t), \ldots, w_{NN}(t), w_{01}(t), \ldots, w_{0N}(t))^T$, $\int_0^t \beta_{ji}(s)ds = w_{ji}(t)$, $i, j = 0, 1, \ldots, N$.

Obviously, the resulting system in (4) is a stochastic system. Hence, we first define leader-following cluster consensus in the strong mean-square sense, as follows.

Definition 1: The system in (1) achieves leader-following cluster consensus in the strong mean-square sense if there exists a protocol $\{u_i, i = 1, \ldots, N\}$ such that the closed-loop system satisfies

$$\lim_{t \to \infty} E(x_i - c_i x_0)^2 = 0, \quad i = 1, \ldots, N$$

where c_i is a constant, determined by communication interactions among agents.

Definition 1 implies that in the nontrivial case ($x_0 \neq 0$), the states of the followers converge to $c_i x_0$ in the strong mean-square sense. This means that as time goes on the states of followers are closely related to the communication interactions among agents.

Note that traditional leader-following consensus and bipartite consensus can be viewed as special cases of leader-following cluster consensus, respectively, with respect to $c_i \equiv 1$ and $c_i = \pm 1, i = 1, \ldots, N$. If $c_i \equiv 1, i = 1, \ldots, N$, then Definition 1 degenerates to the notion of leader-follower consensus in [28]; if $c_i = \pm 1, i = 1, \ldots, N$, Definition 1 degenerates to the notion of leader-following bipartite consensus in [34].

For analysis, the following assumptions are required.

(O_1) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$ contains a spanning tree.
(O_2) $\int_0^\infty f(s)ds = \infty$.
(O_3) $\int_0^\infty f^2(s)ds < \infty$.

Note here that the assumptions made for $f(t)$ in (O_2) and (O_3) are widely adopt in stochastic approximation theory.

III. Main Results

In this section, we study leader-following cluster consensus in the strong mean-square sense based on three types of weighted cooperative–competitive networks, i.e., interactively balanced, sub-balanced, and unbalanced networks, respectively.

A. Interactively Balanced Network

Theorem 1: Under protocol (3), the system in (1) achieves leader-following cluster consensus in the strong mean-square sense if (O_1)–(O_3) hold and $\mathcal{G}$ is interactively balanced.

Proof: Since $\mathcal{G}$ is interactively balanced and (O_1) holds, by Lemma 1, there exists $T = \mathrm{diag}(T_{00}, T_{01}, \dots, T_{0N})$ such that

$$T^{-1}LT = \begin{pmatrix} 0 & \mathbf{0} \\ -G_s \cdot \mathbf{1}_N & L^s_{\mathcal{G}_f} + G_s \end{pmatrix} \triangleq L_s \tag{5}$$

where $L^s_{\mathcal{G}_f} = (l_{sij})$, $l_{sii} = \sum_{j\in\mathcal{N}_i} a_{ij}$, $l_{sij} = -a_{ij}$, $i, j = 1, \dots, N$. Clearly, L_s is a standard Laplacian matrix. By (O_1) and Lemma 2, $-L^s_{\mathcal{G}_f} - G_s$ is Hurwitz. Hence

$$-\left(L^s_{\mathcal{G}_f} + G_s\right)^T B - B\left(L^s_{\mathcal{G}_f} + G_s\right) = -I_N \tag{6}$$

has a positive-definite solution B.

Define $\delta(t) = (\delta_1(t), \dots, \delta_N(t))^T$ with $\delta_i(t) = T_{00}T_{0i}^{-1}x_i(t) - x_0(t)$, $i = 1, \dots, N$. Then, by (4) and (5), one obtains that

$$d\delta = -f(t)\left(L^s_{\mathcal{G}_f} + G_s\right)\delta dt + T_{00}f(t)\overline{T}^{-1}QdW^* \tag{7}$$

in which $\overline{T} = \mathrm{diag}(T_{01}, \dots, T_{0N})$. For convenience, one can assume $W(t) = (W_1(t), \dots, W_N(t))^T$, in which

$$W_i(t) = \begin{cases} w_{0i}(t), & g_i^2 + \sum_{j\in\mathcal{N}_i} T_{00}^2 T_{0i}^{-2} a_{ij}^2 d_{ij}^2 = 0 \\ \dfrac{g_i w_{0i}(t) + \sum_{j\in\mathcal{N}_i} T_{00}T_{0i}^{-1} a_{ij} d_{ij} w_{ji}(t)}{\sqrt{g_i^2 + \sum_{j\in\mathcal{N}_i} T_{00}^2 T_{0i}^{-2} a_{ij}^2 d_{ij}^2}}, & \text{else.} \end{cases}$$

This combining with (7) gives

$$d\delta(t) = -f(t)\left(L^s_{\mathcal{G}_f} + G_s\right)\delta(t)dt + f(t)DdW \tag{8}$$

in which $D = \mathrm{diag}(\sqrt{g_1^2 + \sum_{j\in\mathcal{N}_1} T_{00}^2 T_{01}^{-2} a_{1j}^2 d_{1j}^2}, \dots, \sqrt{g_N^2 + \sum_{j\in\mathcal{N}_N} T_{00}^2 T_{0N}^{-2} a_{Nj}^2 d_{Nj}^2})$. Consider a Lyapunov function $V(t) = \delta^T(t)B\delta(t)$, where B is the positive-definite solution to (6). Hence, by (8) and Itô formula, one obtains

$$dV(t) = -f(t)\delta^T(t)\delta(t)dt + f^2(t)\mathrm{tr}(BDD^T)dt + 2f(t)\delta^T(t)BDdW.$$

Since B is a positive definite

$$dV(t) \le -\frac{f(t)}{\lambda_{\max}(B)}Vdt + f^2(t)\mathrm{tr}\left(BDD^T\right)dt + 2f(t)\delta^T BDdW. \tag{9}$$

Now, one can prove that

$$E\int_{t_0}^t f(s)\delta^T(s)BDdW(s) = 0 \quad \forall 0 \le t_0 \le t.$$

In fact, for any given $t_0 \ge 0$, $T \ge t_0$, define $\tau_m^{t_0} = \{t \ge t_0 | \delta^T(t)B\delta(t) \ge m\}$, where $m > 0$ and then by (9), one gets

$$\begin{aligned} &E\left[V\left(t \wedge \tau_m^{t_0}\right)\chi_{\left\{t \le \tau_m^{t_0}\right\}}\right] - E[V(t_0)] \\ &\le -\frac{1}{\lambda_{\max}(B)}\int_{t_0}^t f(s)E\left[V\left(s \wedge \tau_m^{t_0}\right)\chi_{\left\{s \le \tau_m^{t_0}\right\}}\right]ds \\ &\quad + tr\left(BDD^T\right)\int_{t_0}^t f^2(s)ds \\ &\le tr\left(BDD^T\right)\int_{t_0}^T f^2(s)ds, \quad \forall t_0 \le t \le T. \end{aligned} \tag{10}$$

This means that $\exists \Upsilon_{t_0,T} > 0$ related to t_0 and T satisfying

$$E\left[V\left(t \wedge \tau_m^{t_0}\right)\chi_{\left\{t \le \tau_m^{t_0}\right\}}\right] \le \Upsilon_{t_0,T}, \quad \forall 0 \le t_0 \le T.$$

Note that $t \wedge \tau_m^{t_0} \overrightarrow{a.s.}\, t$ as $m \to \infty$. From the above inequality and the Fatou lemma, one sees that $\sup_{t_0 \le t \le T} E[V(t)] \le \Upsilon_{t_0,T}$. Therefore, $E\int_{t_0}^t f^2(s)V(s)ds \le \sup_{t_0 \le t \le T} E[V(t)]\int_{t_0}^T f^2(s)ds < \infty$, $\forall t_0 \le t \le T$. Since T is arbitrary, one obtains $E\int_{t_0}^t f^2(s)V(s)ds < \infty$, $\forall 0 \le t_0 \le t$. This together with

$$E\int_{t_0}^t f^2(s)\left\|\delta^T(s)BD\right\|^2 ds \le \|B\|\|D\|^2 E\int_{t_0}^t f^2(s)V(s)ds$$

proves $E\int_{t_0}^t f(s)\delta^T(s)BDdW(s) = 0$. From (9), it is seen that $\forall t \ge 0$ and $\Delta \ge 0$

$$\begin{aligned} E[V(t+\Delta)] - E[V(t)] \le& -\frac{1}{\lambda_{\max}(B)}\int_t^{t+\Delta} f(s)E[V(s)]ds \\ &+ tr\left(BDD^T\right)\int_t^{t+\Delta} f^2(s)ds \end{aligned}$$

or

$$\begin{aligned} &\frac{E[V(t+\Delta)] - E[V(t)]}{\Delta} \\ &\le \frac{-\frac{1}{\lambda_{\max}(B)}\int_t^{t+\Delta} f(s)E[V(s)]ds + tr\left(BDD^T\right)\int_t^{t+\Delta} f^2(s)ds}{\Delta}. \end{aligned}$$

Thus

$$\begin{aligned} &\lim_{\Delta \to 0^+} \sup \frac{E[V(t+\Delta)] - E[V(t)]}{\Delta} \\ &\le -\frac{1}{\lambda_{\max}(B)}f(t)E[V(t)] + tr\left(BDD^T\right)f^2(t). \end{aligned}$$

In light of the comparison lemma [37], one has

$$\begin{aligned} E[V(t)] \le& E[V(0)]\exp\{-\frac{1}{\lambda_{\max}(B)}\int_0^t f(s)ds\} \\ &+ tr\left(BDD^T\right)\int_0^t f^2(s)\exp\left\{-\frac{1}{\lambda_{\max}(B)}\int_s^t f(\tau)d\tau\right\}ds, \\ &\forall t \in [0, t+\Delta]. \end{aligned} \tag{11}$$

By (O_3), $\forall \varepsilon > 0$, $\exists s_0 > 0$ satisfying $\int_{s_0}^\infty f^2(s)ds < \varepsilon$. Then

$$\begin{aligned} &tr\left(BDD^T\right)\int_0^t f^2(s)\exp\left\{-\frac{1}{\lambda_{\max}(B)}\int_s^t f(\tau)d\tau\right\}ds \\ &= tr\left(BDD^T\right)\int_0^{s_0} f^2(s)\exp\left\{-\frac{1}{\lambda_{\max}(B)}\int_s^t f(\tau)d\tau\right\}ds \\ &\quad + tr\left(BDD^T\right)\int_{s_0}^t f^2(s)\exp\left\{-\frac{1}{\lambda_{\max}(B)}\int_s^t f(\tau)d\tau\right\}ds \\ &\le tr\left(BDD^T\right)\exp\left\{-\frac{1}{\lambda_{\max}(B)}\int_{s_0}^t f(\tau)d\tau\right\}\int_0^{s_0} f^2(s)ds \\ &\quad + tr\left(BDD^T\right)\int_{s_0}^t f^2(s)ds \\ &\le tr\left(BDD^T\right)\exp\left\{-\frac{1}{\lambda_{\max}(B)}\int_{s_0}^t f(\tau)d\tau\right\}\int_0^\infty f^2(s)ds \\ &\quad + tr\left(BDD^T\right)\int_{s_0}^\infty f^2(s)ds \\ &\le o(1) + tr\left(BDD^T\right)\varepsilon, \quad t \to \infty. \end{aligned} \tag{12}$$

Further, one has

$$\lim_{t\to\infty} tr\left(BDD^T\right)\int_0^t f^2(s)\exp\left\{-\frac{1}{\lambda_{\max}(B)}\int_s^t f(\tau)d\tau\right\}ds = 0. \tag{13}$$

By (O_2), (11), and (13), one obtains that $\lim_{t\to\infty} E[V(t)] = 0$. Note that $\|\delta(t)\|^2 \le ([V(t)]/[\lambda_{\min}(B)])$. This leads to $\lim_{t\to\infty} E\|\delta(t)\|^2 = 0$. Let $c_i = T_{00}^{-1}T_{0i}$, $i = 1,\dots,N$. Then by Definition 1, the system in (1) achieves leader-following cluster consensus in the strong mean-square sense. ■

Theorem 1 shows that under interactively balanced network and measurement noise, protocol (3) drives the states of the followers in (1) to $c_i x_0$ (x_0 is the leader's state) in the strong mean-square sense if assumptions (O_1)–(O_3) hold. Among these assumptions, (O_1) is the requirement on communication topology, and (O_2) and (O_3) are the requirements on consensus gain $f(t)$ for guaranteeing a strong mean-square leader-following cluster consensus.

From Theorem 1, we find that $c_i = T_{00}^{-1}T_{0i}$ in Definition 1 is determined by Laplacian $\mathcal{L}$. Thus, it heavily relies on the communication network but has no relation to $X(0)$.

Remark 2: $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$ reduces to the traditional nonnegative digraph if $d_{ij} \equiv 1 (j \in \mathcal{N}_i)$. In this case, $\mathcal{G}$ is naturally interactively balanced and $T_{0i} \equiv 1$ $(i = 1,\dots,N)$. Thus, Theorem 1 degenerates to [28, Th. 1].

Remark 3: With the effect of measurement noise, the resulting system is essentially a stochastic differential equation, failing the protocols in [18]–[22] and [25] which do not explicitly consider measurement noise. To meet this challenge, our protocol is in the stochastic approximation fashion, using a time-varying function $f(t)$ to eliminate the effect of measurement noise.

Without measurement noise, protocol (3) can be expressed as

$$u_i(t) = f(t)\sum_{j\in\mathcal{N}_i}\left[a_{ij}(d_{ij}x_j - x_i) + g_i(d_{i0}x_0 - x_i)\right] \tag{14}$$

which then yields the following error system:

$$\dot{\delta}(t) = -f(t)\left(L^s_{\mathcal{G}_f} + G_s\right)\delta(t). \tag{15}$$

Theorem 2: Under protocol (14) and $n_{ji}(t) \equiv 0$, the system in (1) achieves leader-following cluster consensus if (O_1) and (O_2) hold and $\mathcal{G}$ is interactively balanced.

Proof: Since $\mathcal{G}$ is interactively balanced and (O_1) holds, taking the same procedures as in Theorem 1, by (15), one has

$$\frac{dV}{dt} \le -\frac{f(t)}{\lambda_{\max}(B)}V(t). \tag{16}$$

Thus, $V(t) \le V(0)\exp\{-(1/[\lambda_{\max}(B)])\int_0^t f(s)ds\}$. By (O_2), $\lim_{t\to\infty} V(t) = 0$. This together with $\|\delta(t)\|^2 \le ([V(t)]/[\lambda_{\min}(B)])$ implies that $\lim_{t\to\infty}\|\delta(t)\| = 0$. ■

Remark 4: Assumption (O_2) naturally holds for $f(t) \equiv 1$. In this case, Theorem 2 reduces to the conclusion that "leader-following cluster consensus is reached if $\mathcal{G}$ is interactively balanced and contains a spanning tree."

B. Interactively Sub-Balanced Network

The following lemma is first given.

Lemma 3: If $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$ is interactively sub-balanced and has a spanning tree, then there exists a nonsingular matrix S such that $SLS^{-1} = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & L_2 \end{pmatrix}$, where all eigenvalues of L_2 are positive.

Proof: Since $\mathcal{G}$ is interactively sub-balanced, by definition, $\mathcal{G}$ does not have directed cycles. Thus, by [15], one can rearrange all nodes in $\mathcal{G}$ such that $i < j$ for $(i,j) \in \mathcal{E}$, and $a_{ij} = 0\ \forall j \ge i$. Therefore, L is a lower triangular matrix, i.e.,

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -g_1 d_{10} & g_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -g_N d_{N0} & -a_{N1}d_{N1} & \cdots & g_N + \sum_{j\in\mathcal{N}_N} a_{Nj} \end{pmatrix}.$$

Since $\mathcal{G}$ has a spanning tree, L has a simple eigenvalue 0, and other eigenvalues are positive. Suppose that

$$S = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_N & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{17}$$

where $a_1 = d_{10}$, $a_i = (g_i d_{i0} + \sum_{j\in\mathcal{N}_i} a_{ij}d_{ij}a_j)/(g_i + \sum_{j\in\mathcal{N}_i} a_{ij})$, $1 \le j < i$, $i = 2,\dots,N$. Then $-g_1 d_{10} + g_1 a_1 = -g_1 d_{10} + g_1(g_1 d_{10}/g_1) = 0, \dots, -g_N d_{N0} - \sum_{j\in\mathcal{N}_N} a_{Nj}d_{Nj}a_j + (g_N + \sum_{j\in\mathcal{N}_N} a_{Nj})a_N = 0$. Therefore

$$SLS^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & g_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -a_{N1}d_{N1} & \cdots & g_N + \sum_{j\in\mathcal{N}_N} a_{Nj} \end{pmatrix} \triangleq \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & L_2 \end{pmatrix}. \tag{18}$$

This implies that all eigenvalues of L_2 are positive. ■

Theorem 3: Under protocol (3), the system in (1) achieves leader-following cluster consensus in the strong mean-square sense if (O_1)–(O_3) hold and $\mathcal{G}$ is interactively sub-balanced.

Proof: By Lemma 3, one can assume that $\eta(t) \triangleq (\eta_1(t), \eta_2(t), \dots, \eta_N(t))^T$, where $\eta_i(t) = x_i(t) - a_i x_0$, $i = 1,2,\dots,N$. Denote $Y(t) \triangleq (x_0, \eta^T(t))^T$. Then, $Y(t) = S\begin{pmatrix} x_0, \\ X(t) \end{pmatrix}$. By (2) and (4), one gets

$$dY(t) = -f(t)SLS^{-1}Y(t)dt + f(t)S\begin{pmatrix} 0 \\ Q \end{pmatrix}M(t)dt.$$

This together with (18) gives

$$d\eta(t) = -f(t)L_2\eta(t)dt + f(t)QdW^*. \tag{19}$$

From Lemma 3, one sees that all eigenvalues of matrix L_2 are positive. Therefore, $-L_2$ is a stable matrix. Adopting the same procedures after (8) in Theorem 1, we get $\lim_{t\to\infty} E||\eta(t)||^2 = 0$. Let $c_i = a_i$, $i = 1,\dots,N$. Then by Definition 1, the system in (1) achieves leader-following cluster consensus in the strong mean-square sense. ■

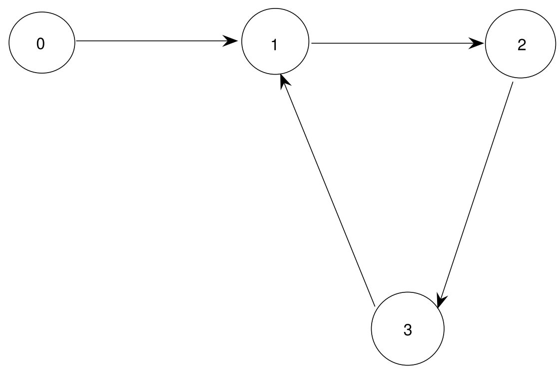

Fig. 1. Communication topology $\mathcal{G}_1$.

Without measurement noise, (19) degenerates to

$$\dot{\eta}(t) = -f(t)L_2\eta(t) \tag{20}$$

where leader-following cluster consensus can still be achieved.

Theorem 4: Under protocol (14) and $n_{ji}(t) \equiv 0$, the system in (1) achieves leader-following cluster consensus if (O_1) and (O_2) hold and $\mathcal{G}$ is interactively sub-balanced.

Proof: Similar to the analysis of Theorem 2, and hence the proof is omitted. ■

Remark 5: Compared with the interactively balanced network, which is supposed to be like the structurally balanced one in a signed digraph, the interactively sub-balanced network is newly defined, first introduced in [25]. Theorems 3 and 4 indicate that leader-following cluster consensus is also reached under an interactively sub-balanced network regardless of measurement noise.

C. Interactively Unbalanced Network

From the above two sections, we can obtain that based on the proposed protocols, leader-following cluster consensus can be achieved for interactively balanced and sub-balanced networks either with or without measurement noise. However, these results do not always hold for an interactively unbalanced network.

Consider a counterexample of an MAS composed of three robots and one leader robot. Their kinematic equations are simplified to (1) with $N = 3$, where $x_i(t)$ is the information state of the ith robot [38]. Information exchange among robots are expressed by $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{A}_1, \mathcal{D}_1)$ with $\mathcal{V}_1 = \{0, 1, 2, 3\}$, $\mathcal{A}_1 = (a_{ij}), a_{10} = a_{21} = a_{32} = 2, a_{13} = 1$, $\mathcal{D}_1 = (d_{ij}), d_{10} = 2, d_{21} = -3, d_{32} = 1, d_{13} = -2$, as shown in Fig. 1. Apparently, $\mathcal{G}_1$ has a spanning tree and contains a directed cycle composed of robots 1–3. Notice that $d_{21}d_{32}d_{13} = 6 \neq 1$. Thus, $\mathcal{G}_1$ is interactively unbalanced.

Take $f(t) = 5/(t+1)$. Then, (O_2) and (O_3) hold. Applying protocol (3) to (1), one obtains that the states of robots are divergent as shown in Fig. 2. Protocol (3) does not solve the leader-following cluster consensus under an interactively unbalanced network. This prompts us to investigate how to design protocols for an interactively unbalanced network to achieve leader-following cluster consensus under measurement noise. We find that pinning control can be a useful tool to solve this challenge.

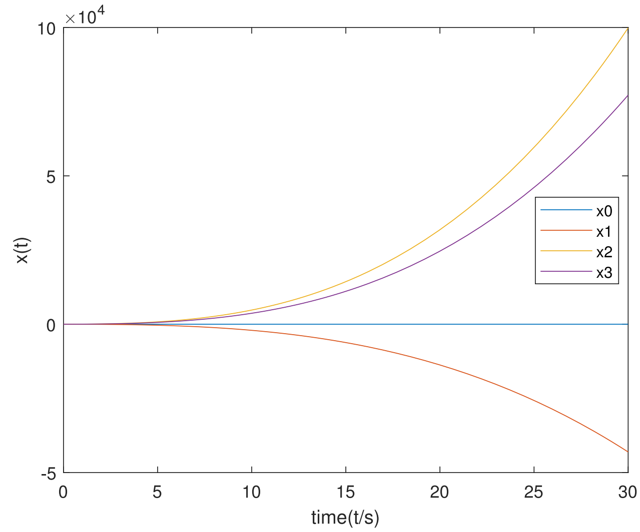

Fig. 2. Evolution of robots' states under interactively unbalanced network $\mathcal{G}_1$.

When $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$ has a spanning tree, without loss of generality, one can assume that the spanning tree is $S_T = (\mathcal{V}, \mathcal{E}_T)$ with $\mathcal{E}_T \subset \mathcal{E}$ and renumber nodes in $\mathcal{V}$ with $i < j$ for $(i, j) \in \mathcal{E}_T$. In S_T, let the directed path from 0 to i be $\mathcal{P}_{0i}^{S_T}(i \neq 0)$. Then, $\mathcal{P}_{0i}^{S_T}$ is unique for any $i \in \mathcal{V}$. Let R_{0i} be the weight product of $\mathcal{P}_{0i}^{S_T}$. Assume that $M = \mathrm{diag}(R_{00}, R_{01}, \ldots, R_{0N})$ with $R_{00} = 1$. Therefore, $M^{-1}LM = L_M$, where $L_M = (l_{M,ij}) \in \mathbb{R}^{(N+1)\times(N+1)}$ and its elements satisfy $l_{M,ii} = \sum_{k\in\mathcal{N}_i} a_{ik} + g_i$

$$l_{M,ij} = \begin{cases} -R_{0i}^{-1}a_{ij}d_{ij}R_{0j}, & (j, i) \in \mathcal{E}\backslash\mathcal{E}_T \\ -a_{ij}, & \text{others.} \end{cases}$$

If $\mathcal{G}$ is interactively unbalanced, by definition, $\mathcal{G}$ has directed cycles and $\exists C_i \neq 1$ in $\hat{\mathcal{G}}$. These facts imply that there exists at least one $(j, i) \in \mathcal{E}\backslash\mathcal{E}_T$ such that $R_{0i}^{-1}d_{ij}R_{0j} \neq 1$. This feature consequently means that the Laplacian of $\mathcal{G}$ is not similar to the standard Laplacian and hence not that useful for convergence analysis, as seen earlier for interactively balanced and sub-balanced networks. Therefore, the key idea in designing protocols for interactively unbalanced networks is to eliminate the effects of $R_{0i}^{-1}d_{ij}R_{0j} \neq 1$. That is why pinning control is employed here.

Theorem 5: Under protocol (3) and the added pinning control, the system in (1) achieves leader-following cluster consensus in the strong mean-square sense if (O_1)–(O_3) hold and $\mathcal{G}$ is interactively unbalanced.

Proof: Since $\mathcal{G}$ is interactively unbalanced and (O_1) holds, there exists $(j, i) \in \mathcal{E}\backslash\mathcal{E}_T$ such that $R_{0i}^{-1}d_{ij}R_{0j} \neq 1$. Assume $\Theta_1 = \{i \mid (j, i) \in \mathcal{E}\backslash\mathcal{E}_T, R_{0i}^{-1}d_{ij}R_{0j} \neq 1\}$. Then, a pinning control is added for agents in Θ_1, i.e.,

$$\begin{aligned} u_i(t) &= f(t)\sum_{j\in\mathcal{N}_i}\big[a_{ij}(d_{ij}x_{ji}(t) - x_i(t)) \\ &\quad + g_i(d_{i0}x_{0i}(t) - x_i(t))\big] + v_i^*(t), \quad i \in \Theta_1 \\ u_i(t) &= f(t)\sum_{j\in\mathcal{N}_i}\big[a_{ij}(d_{ij}x_{ji}(t) - x_i(t)) \\ &\quad + g_i(d_{i0}x_{0i}(t) - x_i(t))\big], \quad i \notin \Theta_1 \end{aligned} \tag{21}$$

where $v_i^*(t)$ is the pinning control.

Due to the definition of interactively unbalanced graph, we obtain that $\exists C_i \neq 1$ in $\hat{\mathcal{G}}$. According to the weight products of directed circles in $\hat{\mathcal{G}}$, the proof will be divided into two cases.

Case 1: In $\hat{\mathcal{G}}$, all $C_i > 0$ and there exists at least one $C_i \neq 1$. We design the pinning control as follows:

$$v_i^*(t) = -k_i f(t) x_i(t), \quad i \in \Theta_1 \tag{22}$$

where $k_i = -\sum_{k\in\mathcal{N}_i} a_{ik} - \sum_{j\in\mathcal{N}_i} l_{M,ij}$. Apply protocol (22) to (21) and assume $\delta_i(t) = R_{00}R_{0i}^{-1}x_i(t) - x_0$. Then

$$d\delta = -f(t)\big(L_M^s + G_s\big)\delta dt + f(t)\mathrm{diag}\Big(R_{01}^{-1}, \ldots, R_{0N}^{-1}\Big) \times QdW^*(t)$$

where $L_M^s = (l_{ij}^s)$

$$l_{ij}^s = \begin{cases} \sum_{j\in\mathcal{N}_i} a_{ij}|R_{0i}^{-1}d_{ij}R_{0j}|, & i = j \\ -a_{ij}|R_{0i}^{-1}d_{ij}R_{0j}|, & i \neq j. \end{cases}$$

Since (O_2) and (O_3) hold, following the proofs after (7), one obtains that the system in (1) achieves leader-following cluster consensus in the strong mean-square sense.

Case 2: In $\hat{\mathcal{G}}$, there exists at least one $C_i < 0$. Define $\Theta_2 = \{j|(j,i) \in \mathcal{E}\backslash\mathcal{E}_T, R_{0i}^{-1}d_{ij}R_{0j} < 0\}$. We design the pinning control as follows:

$$v_i^*(t) = -k_{ii}f(t)x_i(t) + f(t)\sum_{j\in\mathcal{N}_i\cap\Theta_2} k_{ij}x_{ji}(t), \quad i \in \Theta_1 \tag{23}$$

where $k_{ii} = -\sum_{k\in\mathcal{N}_i} a_{ik} + \sum_{j\in\mathcal{N}_i} |l_{M,ij}|$, $k_{ij} = -a_{ij}d_{ij}[1 + \mathrm{sgn}(l_{M,ij})]$. Apply protocol (23) to (21) and assume $\delta_i(t) = R_{00}R_{0i}^{-1}x_i(t) - x_0$. Then

$$d\delta = -f(t)\big(L_M^s + G_s\big)\delta dt + f(t)\mathrm{diag}\Big(R_{01}^{-1}, \ldots, R_{0N}^{-1}\Big) \times \widetilde{Q}dW^*(t).$$

Similarly, one obtains that the system in (1) achieves leader-following cluster consensus in the strong mean-square sense. ■

In the absence of measurement noise, we have the following conclusion.

Theorem 6: Under protocol (21)–(23) and $n_{ji}(t) \equiv 0$, the system in (1) achieves leader-following cluster consensus if (O_1) and (O_2) hold and $\mathcal{G}$ is interactively unbalanced.

Remark 6: Compared with the conditions in Theorems 2, 4, and 6 with $n_{ji}(t) \equiv 0$, Assumption (O_3) plays an important role in Theorems 1, 3, and 5 with measurement noise. Actually, (O_3) is introduced to eliminate the harmful effects of measurement noise.

Remark 7: Different from the previous results on cluster consensus or leader-following cluster consensus of MASs perturbed by measurement noise, e.g., [35] and [36], where it is assumed that the communication interactions are described by binary cooperative–competitive network, in this present work, we consider the weighted cooperative–competitive network, which includes binary cooperative–competitive network as its special case. Such network is more closer to the practice, where like/dislike and agreement/disagreement between agents are not merely an ON/OFF signal.

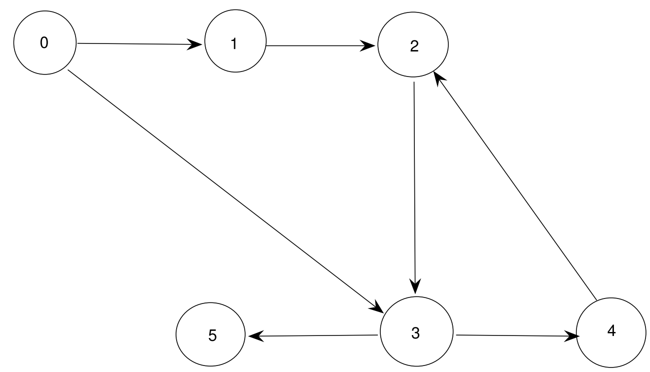

Fig. 3. Communication topology $\mathcal{G}_2$.

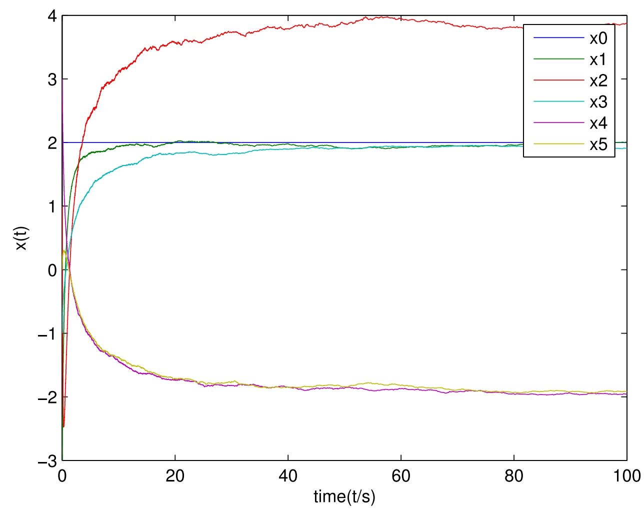

Fig. 4. Evolution of vehicles' states under $\mathcal{G}_2$ with measurement noise.

IV. Numerical Example

Example 1: Consider a group of one leader and five vehicles moving together, where vehicles are described by a simplified kinematic (1) with $N = 5$ [38]. The basic idea is to drive the state of each vehicle toward the multiple of the leader's state. Fig. 3 shows the information exchanges among vehicles, expressed by $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{A}_2, \mathcal{D}_2)$, where $\mathcal{V}_2 = \{0, 1, 2, 3, 4, 5\}$, $\mathcal{A}_2 = (a_{ij})$, $a_{10} = a_{21} = 2$, $a_{30} = a_{32} = 1$, $a_{24} = a_{53} = 3$, $a_{43} = 4$, and $\mathcal{D}_3 = (d_{ij})$, $d_{10} = d_{30} = 1$, $d_{21} = 2$, $d_{24} = -2$, $d_{32} = 0.5$, $d_{43} = d_{53} = -1$. Clearly, $\mathcal{G}_2$ has a spanning tree, i.e., (O_1) holds. Notice that $d_{32}d_{43}d_{24} = 1$, $d_{10}d_{21}d_{32}d_{30}^{-1} = 1$ and $d_{10}d_{21}(d_{24}d_{43}d_{30})^{-1} = 1$. Therefore, $\mathcal{G}_2$ is interactively balanced. Take $f(t) = 1/(t + 1)$ in protocol (3). Thus, (O_2) and (O_3) hold. Assume $x_0 = 2$ and $X(0) = (-3, 1, -1, 3, 0)^T$. Then, Fig. 4 shows the state trajectories of vehicles. Apparently, as time goes on, vehicles are divided into three clusters: 1) leader 0 and vehicles 1 and 3; 2) vehicles 4 and 5; and 3) vehicle 2. This means that vehicles achieve leader-following cluster consensus in the strong mean-square sense, which is consistent with Theorem 1.

Without measurement noise, we choose $f(t) = t$ in protocol (14), then (O_2) holds. Fig. 5 shows that leader-following cluster consensus is reached under $\mathcal{G}_2$. This verifies the effectiveness of Theorem 2.

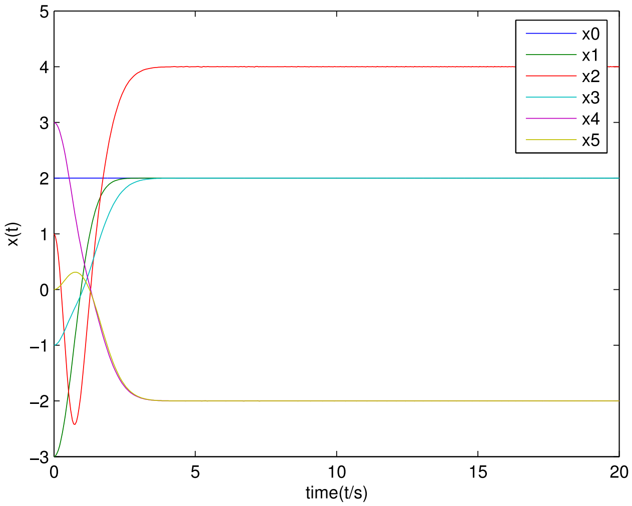

Fig. 5. Evolution of vehicles' states under $\mathcal{G}_2$ without measurement noise.

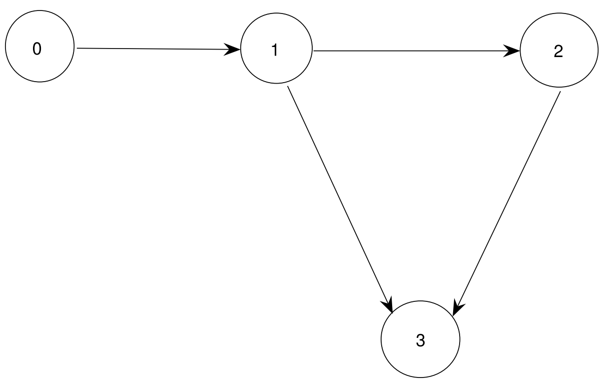

Fig. 6. Communication topology $\mathcal{G}_3$.

Example 2: In this example, we consider one leader and three vehicles with dynamics (1). Communications among vehicles are described by $\mathcal{G}_3 = (\mathcal{V}_3, \mathcal{E}_3, \mathcal{A}_3, \mathcal{D}_3)$, as shown in Fig. 6, where $\mathcal{V}_3 = \{0, 1, 2, 3\}$, $\mathcal{A}_3 = (a_{ij})$, $a_{10} = a_{21} = 2$, $a_{31} = 1$, $a_{32} = 3$, and $\mathcal{D}_1 = (d_{ij}), d_{10} = 2, d_{21} = d_{32} = 1, d_{31} = -1$. It is direct to check that $\mathcal{G}_3$ is interactively sub-balanced and has a spanning tree. We choose $f(t) = 1/(t+1)$ in protocol (3). Then, (O_2) and (O_3) hold. We adopt the initial conditions of $x_0 = 1$ and $X(0) = (-1, 4, 0)^T$. The state trajectories of vehicles in Fig. 7 show that vehicles are divided into two clusters: leader 0 and vehicle 3; and vehicles 1 and 2, respectively. Therefore, leader-following cluster consensus in the strong mean-square sense is achieved for an interactively sub-balanced network. This verifies the result of Theorem 3.

When there is no measurement noise, one can choose $f(t) = t$ in protocol (14). Leader-following cluster consensus is observed in Fig. 8. This is in line with the result in Theorem 4.

Example 3: As is shown in Fig. 2, for an interactively unbalanced network $\mathcal{G}_1$, the MAS with protocol (3) may be divergent. To achieve leader-following cluster consensus under measurement noise, a pinning control (21), (22) is added since all $C_i > 0$ and $C_i \neq 1$ in $\hat{\mathcal{G}}_1$. Therefore, it follows from Theorem 5 that $k_1 = 5$ in (22). Apply (21) and (22) to the

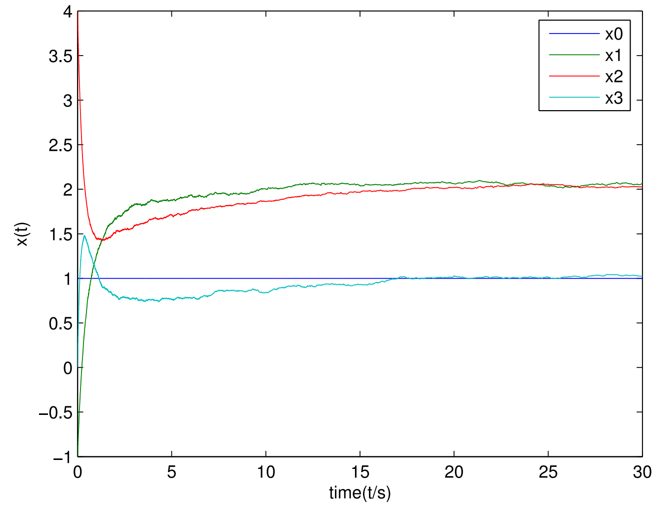

Fig. 7. Evolution of vehicles' states under $\mathcal{G}_3$ with measurement noise.

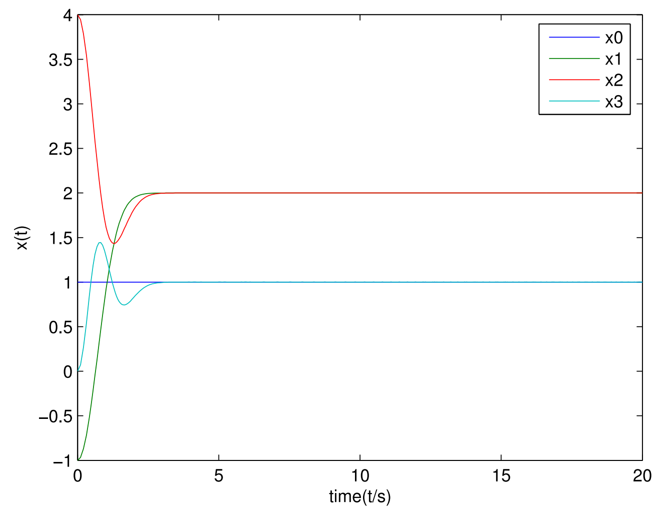

Fig. 8. Evolution of vehicles' states under $\mathcal{G}_3$ without measurement noise.

robots. Then, the trajectories of the states of the three robots under $\mathcal{G}_1$ are depicted in Fig. 9, from which it can be seen that leader-following cluster consensus for an interactively unbalanced network with pinning control is indeed achieved.

When there is no measurement noise, we apply protocols (21) and (22) to the three robots. Fig. 10 shows that leader-following cluster consensus is reached under an interactively unbalanced network. This verifies the effectiveness of Theorem 6.

V. Conclusion

This work investigates leader-following cluster consensus of MASs subject to measurement noise, under three types of weighted cooperative–competitive networks, namely, interactively balanced, sub-balanced, and unbalanced networks. For the former two cases, a distributed protocol with a time-varying gain function is proposed, under which it is proved that leader-following cluster consensus in the strong mean-square sense can be reached. For the third case, the strong mean-square leader-following cluster consensus is achieved with

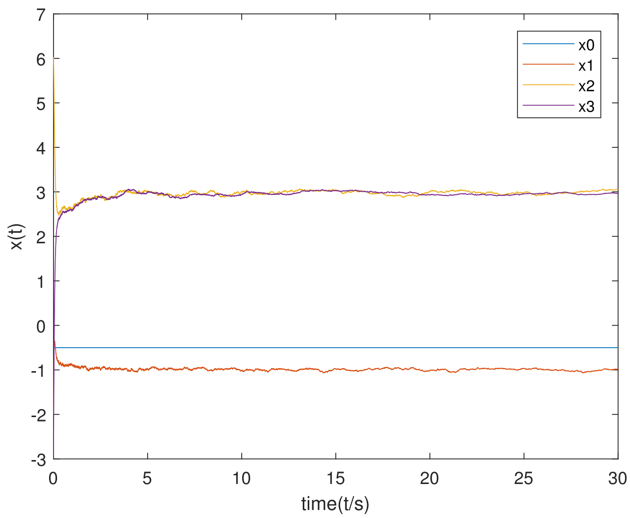

Fig. 9. Evolution of robots' states with pinning control under $\mathcal{G}_1$.

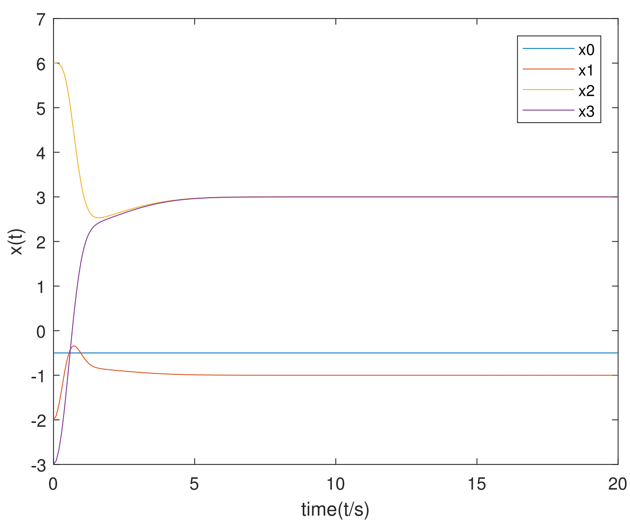

Fig. 10. Evolution of robots' states with pinning control and $n_{ji}(t) \equiv 0$ under $\mathcal{G}_1$.

the aid of pinning control. Furthermore, corresponding results have also been presented for the case without measurement noise.

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